Chapter 2: Feynman–Kac Formula

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Preview

In MATH 5635, we encountered two fundamental approaches for pricing contingent claims: the partial differential equation (PDE) method, exemplified by the Black–Scholes equation, and the risk-neutral valuation method, which prices claims as expected discounted payoffs under a risk-neutral probability measure. These two perspectives already suggest a deep connection between stochastic processes and partial differential equations. This chapter makes this connection precise by exploring the interplay between stochastic calculus and PDE theory. We begin by reviewing the formulation of stochastic differential equations (SDEs) and highlighting their Markovian structure. We then establish a link between SDEs and PDEs through the Feynman–Kac formula, which provides a probabilistic representation of solutions to a broad class of parabolic PDEs. Using the Feynman–Kac formula, we revisit the PDE approach to option pricing and extend it to more general market models beyond the classical Black–Scholes framework.

Key topics in this chapter:

1. Markov property of SDEs;
2. Feynman–Kac formula;
3. Risk-neutral pricing via Feynman–Kac.

1 Stochastic Differential Equations

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and $\{B_t\}_{t\in[0,T]}$ is a standard (1-dimensional) Brownian motion adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$. We first recall the formulation of a stochastic differential equation (SDE), which is a process that satisfies the Itô diffusion

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t,\ \ X_0 = \xi,\ \ 0 \le t \le T, \tag{1}$$

where ξ is a $\mathcal{F}_0$-random variable, and $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Alternatively, we can also express the equation in integral form:

$$X_t = \xi + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s.$$

A ***strong solution*** of (1) is a process $\{X_t\}_{t\in[0,T]}$ that satisfies the Itô diffusion (1) in the given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

1.1 Existence and Uniqueness of Solution

The existence and uniqueness of (strong) solution is given by the following regularity conditions:

Definition 1.1 A function $f(t,x)$ is said satisfy the

1. ***global Lipschitz condition*** if there exists $K > 0$ such that, for any $x, y \in \mathbb{R}$ and $t \geq 0$,
$$|f(t,x) - f(t,y)| \leq K|x-y|;$$
2. ***linear growth condition*** if there exists $L > 0$ such that, for any $t \geq 0$ and $x \in \mathbb{R}$,
$$|f(t,x)| \leq L(1+|x|).$$

The following existence and uniqueness theorem holds under the Lipschitz and linear-growth conditions:

Theorem 1.1 Suppose that the coefficients b and σ satisfy the global Lipschitz and linear growth conditions, and $\mathbb{E}[|\xi|^2] < \infty$. Then, the SDE (1) admits a unique solution, which satisfies
$$\mathbb{E}\left[\int_0^T |X_t|^2\, dt\right] < \infty.$$

1.2 Markov Property

We shall show that conditional expectations of the solution of the SDE (1) satisfies the Markov property. We first recall the Markov property of a stochastic process:

Definition 1.2 Let $\mathcal{T} = \mathbb{R}_+$ or $\mathbb{N}_0$. A $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$-adapted process $X = \{X_t\}_{t\in\mathcal{T}}$ is called a ***Markov process*** if, for any $s \leq t \in \mathcal{T}$ and $A \in \mathcal{F}_t$,
$$\mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|\sigma(X_s)) \text{ a.s.}$$
Equivalent, X is Markov if, for any $s \leq t \in \mathcal{T}$ and any bounded, Borel measurable function f, there exists another Borel measurable function g such that
$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = g(X_s)$$

Given $t \in [0,T]$ and $x \in \mathbb{R}$, we introduce the process $\{X^{t,x}_{t+h}\}_{h\geq 0}$ as the solution of the following SDE:

$$X_{t+h}^{t,x} = x + \int_t^{t+h} b(u, X_u^{t,x})\, du + \int_t^{t+h} \sigma(u, X_u^{t,x})\, dB_u. \tag{2}$$

In other words, $\{X_{t+h}^{t,x}\}_{h\geq 0}$ is the solution to the same SDE (1), except that we specify the initial time t (instead of 0), with initial condition at t given by $X_t^{t,x} = x$.

On the other hand, using the SDE (1) satisfied by X, we have

$$\begin{aligned}
X_{t+h} &= \xi + \int_0^{t+h} b(u, X_u)\, du + \int_0^{t+h} \sigma(u, X_u)\, dB_u \\
&= \left(\xi + \int_0^t b(u, X_u)\, du + \int_0^t \sigma(u, X_u)\, dB_u\right) + \int_t^{t+h} b(u, X_u)\, du + \int_t^{t+h} \sigma(u, X_u)\, dB_u \\
&= X_t + \int_t^{t+h} b(u, X_u)\, du + \int_t^{t+h} \sigma(u, X_u)\, dB_u.
\end{aligned}$$

Note that both processes X_{t+h} and X_{t+h}^{t,X_t} satisfies the same equation for $h \geq 0$. By the uniqueness of solution, we have the following ***flow property***

$$X_{t+h} = X_{t+h}^{t,X_t},\ \ h \geq 0. \tag{3}$$

In particular, $X_t^{0,\xi} = X_t$, $t \in [0,T]$.

We also define the family of (induced) probability measures $\{\mathbb{P}^{t,x}\}_{t\in[0,T],x\in\mathbb{R}}$ such that, under $\mathbb{P}^{t,x}$, the solution of (1) satisfies $X_t = x = X_t^{t,x}$. Intuitively, $\mathbb{P}^{t,x}$ describes the law of the stochastic process X when it is initialized at the point x at time t. We also denote the associated expected value of $\mathbb{P}^{t,x}$ by $\mathbb{E}^{t,x}$.

To prove the Markov property, we shall invoke the independence lemma:

Lemma 1.2 Let $X_1, \ldots, X_n$ be $\mathcal{G}$-measurable random variables, and $Y_1, \ldots, Y_m$ be random variables that are independent of $\mathcal{G}$. Let $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a measurable function and define

$$g(x_1, \ldots, x_n) := \mathbb{E}[f(x_1, \ldots, x_n, Y_1, \ldots, Y_m)].$$

Then,

$$\mathbb{E}[f(X_1, \ldots, X_n, Y_1, \ldots, Y_m)|\mathcal{G}] = g(X_1, \ldots, X_n).$$

The following theorem indicates that the solution process $\{X_t\}_{t\in[0,T]}$ of (1) satisfies the Markov property in the following general manner:

Theorem 1.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, Borel-measurable function, and $c : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be bounded and measurable. Then, for any $t, h \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\Big[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h}) \;\Big|\; \mathcal{F}_t\Big] = \mathbb{E}^{t,X_t}\Big[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h})\Big]. \tag{4}$$

Remark 1.4. The RHS of (4) can be understood as follows. Let

$$g(t,x) := \mathbb{E}^{t,x}\Big[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h})\Big] = \mathbb{E}\left[e^{-\int_t^{t+h} c(s,X_s^{t,x})} f(X_{t+h}^{t,x})\right]. \tag{5}$$

Then the right-hand side of (4) is the $\mathcal{F}_t$-measurable random variable $g(t, X_t)$, where

$$g(t,X_t) = \mathbb{E}^{t,X_t}\Big[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h})\Big] = \mathbb{E}\left[e^{-\int_t^{t+h} c(s,X_s^{t,x})} f(X_{t+h}^{t,x})\right]\Bigg|_{x=X_t}.$$

On the other hand, the LHS of (4) is a conditional expectation taken under the original probability measure $\mathbb{P}$, under which the process X_t is defined and starts at time 0 with initial condition $X_0 = \xi$.

Equation (4) expresses the Markov property of the solution X to the SDE. It states that, conditional on the information available up to time t, the future evolution of the process depends on the past only through its current state X_t, which is given by $g(t, X_t)$. In particular, the conditional expectation of $f(X_{t+h})$ given $\mathcal{F}_t$ coincides with the unconditional expectation of $f(X_{t+h})$ for a process started from the state X_t at time t. Consequently, the history of the process prior to time t contains no additional information relevant for predicting its future evolution.

Proof of Theorem 1.3. By the flow property (3), we have

$$X_{t+h} = X_{t+h}^{t,X_t} \qquad \text{a.s.} \tag{6}$$

for any $t, h \geq 0$ and $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$ and $t \geq 0$, the process $\{X_{t+h}^{t,x}\}_{h \geq 0}$ satisfies

$$\begin{aligned} X_{t+h}^{t,x} &= x + \int_t^{t+h} b(u, X_u^{t,x})\,du + \int_t^{t+h} \sigma(u, X_u^{t,x})\,dB_u \\ &= x + \int_0^h b(t+u, X_{t+u}^{t,x})\,du + \int_0^h \sigma(t+u, X_{t+u}^{t,x})\,d\hat{B}_u^t, \end{aligned}$$

where $\hat{B}_u^t := B_{t+u} - B_t$ is a Brownian motion under the filtration $\{\hat{\mathcal{F}}_u := \mathcal{F}_{t+u}\}_{u \geq 0}$, which is independent of $\mathcal{F}_t$.

Thus we can write

$$X_{t+h}^{t,x} = F_{t+h}(t, x, \{\hat{B}_u^t\}_{u \in [0,h]}),$$

where for each $x \in \mathbb{R}$, the randomness of $X^{t,x}_{t+h}$ depends only on the increment $\hat{B}^t$ and is independent of $\mathcal{F}_t$.

Define g as in (5). In particular, using the notation above, we can write

$$g(t,x) := \mathbb{E}\left[e^{-\int_t^{t+h} c(s,F_s(t,x,\{\hat{B}^t_u\}_{u\in[0,s-t]}))\,ds} f(F_{t+h}(t,x,\{\hat{B}^t_u\}_{u\in[0,h]}))\right].$$

Since $\{\hat{B}^t_u\}_{u\geq 0}$ is independent of $\mathcal{F}_t$ and X_t is $\mathcal{F}_t$-measurable, Lemma 1.2 yields

$$\begin{aligned}
&\mathbb{E}^{t,X_t}\left[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h})\right] \\
&= g(t,X_t) \\
&= \mathbb{E}\left[e^{-\int_t^{t+h} c(s,F_s(t,X_t,\{\hat{B}^t_u\}_{u\in[0,s-t]}))\,ds} f(F_{t+h}(t,X_t,\{\hat{B}^t_u\}_{u\in[0,h]})) \;\Big|\; \mathcal{F}_t\right].
\end{aligned}$$

By the flow property (6),

$$F_{t+h}(t,X_t,\{\hat{B}^t_u\}_{u\in[0,h]}) = X^{t,X_t}_{t+h} = X_{t+h} \quad \text{a.s.}$$

Therefore,

$$\mathbb{E}^{t,X_t}\left[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h})\right] = \mathbb{E}\left[e^{-\int_t^{t+h} c(s,X_s)\,ds} f(X_{t+h}) \;\Big|\; \mathcal{F}_t\right] \quad \text{a.s.}$$

□

2 Feynman–Kac Formula

The Feynman–Kac formula connects the solution of SDEs to PDEs. Let X be the solution of (1), and define

$$g(t,x) := \mathbb{E}^{t,x}\left[e^{-\int_t^T c(u,X_u)\,du} f(X_T)\right], \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $c : [0,T] \to \mathbb{R}$ are measurable, $c(\cdot)$ is bounded, and $\mathbb{E}^{t,x}[|f(X_T)|] < \infty$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Then g satisfies a parabolic-type PDE.

To establish the Feynman–Kac formula, we first show the following martingale property.

Lemma 2.1 Let $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be defined as in (7). Then, the process

$$\left\{e^{-\int_0^t c(u,X_u)\,du} g(t,X_t)\right\}_{t\in[0,T]}$$

is a martingale.

Proof. By Theorem 1.3, for any $t \in [0,T]$ and $h := T - t$, we have

$$\mathbb{E}\left[e^{-\int_t^T c(u,X_u)\,du} f(X_T) | \mathcal{F}_t\right] = \mathbb{E}^{t,X_t}\left[e^{-\int_t^T c(u,X_u)\,du} f(X_T)\right] = g(t,X_t).$$

Hence, for any $t \in [0,T]$,

$$e^{-\int_0^t c(u,X_u)\,du} g(t,X_t) = \mathbb{E}\left[e^{-\int_0^T c(u,X_u)\,du} f(X_T) | \mathcal{F}_t\right].$$

Using this and the tower property, for any $0 \le s \le t$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^t c(u,X_u)\,du} g(t,X_t) \big| \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^T c(u,X_u)\,du} f(X_T) | \mathcal{F}_t\right] \big| \mathcal{F}_s\right] \\ &= \mathbb{E}\left[e^{-\int_0^T c(u,X_u)\,du} f(X_T) \big| \mathcal{F}_s\right] \\ &= e^{-\int_0^s c(u,X_u)\,du} g(s,X_s).\end{aligned}$$

□

We now introduce the Feynman–Kac formula which connects PDEs and SDEs:

Theorem 2.2 (Feynman–Kac Formula) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous, and $c : [0,T] \times \mathbb{R} \to \mathbb{R}$ be bounded from below. Let X be the solution of (1). Then the following statements hold:

(i) (**PDE ⇒ stochastic representation**) If $g \in C^{1,2}([0,T) \times \mathbb{R})$ solves

$$\begin{cases} g_t(t,x) + b(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) - c(t,x)\,g(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ g(T,x) = f(x). \end{cases} \tag{8}$$

Then, for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$g(t,x) = \mathbb{E}^{t,x}\left[e^{-\int_t^T c(s,X_s)\,ds} f(X_T)\right]. \tag{9}$$

(ii) (**Stochastic representation ⇒ PDE**) Conversely, define $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ by (9). If $g \in C^{1,2}([0,T) \times \mathbb{R})$, then g is a classical solution of the PDE (8).

The Feynman–Kac formula provides the fundamental link between arbitrage-free pricing under a risk-neutral measure and linear parabolic partial differential equations. In finance, it shows that the discounted expected payoff and the solution of the pricing PDE represent the same value function, thereby justifying both Monte Carlo simulation and PDE-based valuation methods. From a mathematical perspective, it identifies the solution of the PDE as the conditional expectation of a functional of the underlying diffusion, yielding existence, uniqueness, and a probabilistic interpretation of solutions to parabolic equations.

Proof of Theorem 2.2. Suppose that $g \in C^{1,2}([0,T)$ solves (8). Applying Itô's lemma to the process $g(t, X_t)$, yields

$$\begin{aligned} dg(t,X_t) &= g_t(t,X_t)\,dt + g_x(t,X_t)\,dX_t + \frac{1}{2}g_{xx}(t,X_t)d\langle X\rangle_t \\ &= \left(g_t(t,X_t) + b(t,X_t)g_x(t,X_t) + \frac{1}{2}\sigma^2(t,X_t)g_{xx}(t,X_t)\right)\,dt + g_x(t,X_t)\sigma(t,X_t)\,dB_t \\ &= c(t,X_t)g(t,X_t)\,dt + g_x(t,X_t)\sigma(t,X_t)\,dB_t, \end{aligned}$$

where the last line follows from the fact that g satisfies the PDE (8). Now, by the product rule, we further have

$$d\left(e^{-\int_0^t c(s,X_s)\,ds}g(t,X_t)\right) = e^{-\int_0^t c(s,X_s)\,ds}g_x(t,X_t)\sigma(t,X_t)\,dB_t.$$

Integrating the above equations from t to T yields

$$e^{-\int_t^T c(s,X_s)\,ds}g(T,X_T) = g(t,X_t) + \int_t^T e^{-\int_t^s c(u,X_u)\,du}g_x(s,X_s)\sigma(s,X_s)\,dB_s.$$

Taking expectations on both sides under $\mathbb{E}^{t,x}$, under which $X_t = x$, along with the terminal condition in (8), we have

$$g(t,x) = \mathbb{E}^{t,x}\left[e^{-\int_t^T c(s,X_s)\,ds}g(T,X_T)\right] = \mathbb{E}^{t,x}\left[e^{-\int_t^T c(s,X_s)\,ds}f(X_T)\right].$$

To show the converse, we define g as in (9). To see that g satisfies the terminal condition of (8), note that

$$g(T,x) = \mathbb{E}^{T,x}\left[f(X_T)\right] = f(x),$$

since under $\mathbb{E}^{T,x}$, $X_T = x$.

To proceed, by Theorem 1.3, $e^{-\int_0^t c(s,X_s)\,ds}g(t,X_t)$ is a martingale. Hence, for any $t \geq 0$, $h > 0$ with $t + h \leq T$, we have

$$e^{-\int_0^t c(s,X_s)\,ds}g(t,X_t) = \mathbb{E}[e^{-\int_0^{t+h} c(s,X_s)\,ds}g(t+h,X_{t+h})|\mathcal{F}_t],$$

which implies

$$g(t,X_t) = \mathbb{E}[e^{-\int_t^{t+h} c(s,X_s)\,ds}g(t+h,X_{t+h})|\mathcal{F}_t].$$

In particular, under the measure $\mathbb{E}^{t,x}$, we have

$$g(t,x) = \mathbb{E}^{t,x}\left[e^{-\int_t^{t+h} c(s,X_s)\,ds}g(t+h,X_{t+h})\right]$$

On the other hand, by Itô's lemma, for any $h \geq 0$,

$$e^{-\int_t^{t+h} c(s,X_s)\,ds}g(t+h,X_{t+h}) - g(t,X_t)$$

$$= \int_t^{t+h} e^{-\int_t^s c(u,X_u)\,du} \left(-c(s,X_s)g(s,X_s) + g_t(s,X_s) + b(s,X_s)g_x(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)g_{xx}(s,X_s) \right) ds$$
$$+ \int_t^{t+h} e^{-\int_t^s c(u,X_u)\,du} g_x(s,X_s)\sigma(s,X_s)\,dB_s.$$

Hence, by taking expectations with respect to $\mathbb{E}^{t,x}$,

$$\begin{aligned} 0 &= \mathbb{E}^{t,x}\left[e^{-\int_t^{t+h} c(s,X_s)\,ds} g(t+h, X_{t+h}) \right] - g(t,x) \\ &= \mathbb{E}^{t,x}\Bigg[\int_t^{t+h} e^{-\int_t^s c(u,X_u)\,du} \Big(-c(s,X_s)g(s,X_s) + g_t(s,X_s) + b(s,X_s)g_x(s,X_s) \\ &\qquad + \frac{1}{2}\sigma^2(s,X_s)g_{xx}(s,X_s) \Big)\,ds \Bigg]. \end{aligned}$$

Since this holds for any $h \geq 0$, we deduce that the integrand must be zero, and thus we arrive at the PDE (8). □

Example 2.1 The price of a risky asset $\{S_t\}_{t\in[0,T]}$ follows the Black-Scholes model with rate of return μ and volatility σ. Under the risk-neutral probability measure $\widetilde{\mathbb{P}}$, the dynamics of $\{S_t\}_{t\in[0,T]}$ is given by

$$dS_t = S_t r\,dt + S_t\sigma\,d\widetilde{B}_t.$$

Comparing with the general SDE of the form (1), the coefficients are given by $b(t,s) = rs$ and $\sigma(t,s) = \sigma s$.
Consider a contingent claim written on the risky asset with payoff $f(S_T)$ at the time of maturity T. The risk-neutral price of the contingent claim $V(t,s)$ at time t is given by

$$V(t,S_t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t))} f(S_T) | \mathcal{F}_t \right] = \widetilde{\mathbb{E}}^{t,S_t}[e^{-r(T-t)} f(S_T)],$$

where the last equality follows from the Markov property; see Theorem 1.3.
By identifying the discounting $c(t,s) = r$ and using the Feynman–Kac formula, the function

$$V(t,s) = \widetilde{\mathbb{E}}^{t,s}[e^{-r(T-t)} f(S_T)]$$

is the solution of the following PDE:

$$\begin{cases} V_t(t,s) + rV_s(t,s) + \frac{1}{2}\sigma^2 V_{ss}(t,s) - rV(t,s) = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ V(T,s) = f(s), \end{cases}$$

which is precisely the Black-Scholes PDE.

3 Multidimensional Feynman–Kac Formula

The Feynman–Kac formula can be naturally extended to multidimensional SDEs. Let $\boldsymbol{B}_t = (B_t^1, \ldots, B_t^d)$ be a d-dimensional Brownian motion. Consider a m-dimensional process, which is the solution of the following SDE:

$$\begin{cases} d\boldsymbol{X}_t = \boldsymbol{b}(t, \boldsymbol{X}_t)\, dt + \boldsymbol{\sigma}(t, \boldsymbol{X}_t)\, d\boldsymbol{B}_t, \\ \boldsymbol{X}_0 = \boldsymbol{x}_0, \end{cases} \tag{10}$$

where the coefficients are given by $\boldsymbol{b} = (b^1, \ldots, b^m) : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$, $\boldsymbol{\sigma} = (\sigma^{ij})_{1 \le 1 \le m, 1 \le j \le d} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$. We can also express the SDE of $\boldsymbol{X} = (X^1, \ldots, X^m)$ in a component-wise manner: for $i = 1, \ldots, m$,

$$\begin{cases} dX_t^i = b^i(t, X_t^1, \ldots, X_t^m)\, dt + \displaystyle\sum_{j=1}^{d} \sigma^{ij}(t, X_t^1, \ldots, X_t^m)\, dB_t^j, \\ X_0^i = x_0^i. \end{cases}$$

The Markov property and the Feymann–Kac formula can be generalized as follows.

Theorem 3.1 Let $f : \mathbb{R}^m \to \mathbb{R}$ be a bounded, Borel-measurable function, and let $c : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ be bounded and measurable. Then, for any $t, h \ge 0$ and $\boldsymbol{x} \in \mathbb{R}^m$,

$$\mathbb{E}\left[e^{-\int_t^{t+h} c(s, \boldsymbol{X}_s)\, ds} f(\boldsymbol{X}_{t+h}) \,\middle|\, \mathcal{F}_t \right] = \mathbb{E}^{t, \boldsymbol{X}_t}\left[e^{-\int_t^{t+h} c(s, \boldsymbol{X}_s)\, ds} f(\boldsymbol{X}_{t+h}) \right]. \tag{11}$$

To introduce the Feynman–Kac formula, we introduce the following second-order differential operator $\mathcal{L}_t$ by

$$\mathcal{L}_t \varphi(\boldsymbol{x}) := \sum_{i=1}^{m} b^i(t, \boldsymbol{x})\, \partial_{x_i} \varphi(\boldsymbol{x}) + \frac{1}{2} \sum_{i,k=1}^{m} \big(\boldsymbol{\sigma}(t, \boldsymbol{x}) \boldsymbol{\sigma}^\top(t, \boldsymbol{x})\big)_{ik}\, \partial^2_{x_i x_k} \varphi(\boldsymbol{x}), \qquad \varphi \in C^2(\mathbb{R}^m).$$

Theorem 3.2 (Multidimensional Feynman–Kac) Let $f : \mathbb{R}^m \to \mathbb{R}$ be bounded and continuous, and $c : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ be bounded from below. Let $\boldsymbol{X}$ be the solution of the multidimensional SDE (10). Then, the following statements hold:

(i) (**PDE ⇒ stochastic representation**) If $g \in C^{1,2}([0, T) \times \mathbb{R}^m)$ solves

$$\begin{cases} g_t(t, \boldsymbol{x}) + \mathcal{L}_t g(t, \boldsymbol{x}) - c(t, \boldsymbol{x})\, g(t, \boldsymbol{x}) = 0, & (t, \boldsymbol{x}) \in [0, T) \times \mathbb{R}^m, \\ g(T, \boldsymbol{x}) = f(\boldsymbol{x}), \end{cases} \tag{12}$$

then, for all $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^m$,

$$g(t, \boldsymbol{x}) = \mathbb{E}^{t,\boldsymbol{x}}\left[e^{-\int_t^T c(s,\boldsymbol{X}_s)\, ds}\, f(\boldsymbol{X}_T)\right]. \tag{13}$$

(ii) (**Stochastic representation ⇒ PDE**) Conversely, define $g : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ by (13). If $g \in C^{1,2}([0, T) \times \mathbb{R}^m)$, then g is a classical solution of the PDE (12).

In the next chapter, we will connect risk-neutral pricing of Asian options with the multidimensional Feynman–Kac formula.

4 Further Readings

1. Strong Markov property of SDEs;
2. Existence and uniqueness theorem of SDEs (Lipschitz theory);
3. Weak solutions of SDEs;
4. Kolmogorov (forward/backward) equations.