MATH5636 Stochastic Calculus for Finance II Spring 2026

Chapter 3: Pricing of Exotic Options: Asian Options

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Preview

In this and the coming chapters, we will focus on pricing of exotic options. Herein, we first consider Asian options, where the payoff at the time of maturity depends on the average historical movement of the underlying assets. Two types of Asian options are considered – geometric and arithmetic. For geometric type Asian options, we derive the explicit risk-neutral price using probabilistic method and discuss the associated pricing PDE. While no closed-form solutions are available for arithmetic type options, we introduce the method of dimension reduction so that the risk-neutral price can be characterized by a simple parabolic PDE.

Key topics in this chapter:

1. Risk-neutral pricing of Asian options;
2. Kemna—Vorst formula;
3. Change of numéraire.

1 Asian Options

The payoff of an Asian option is *path-dependent*, which depends on the time average of the price of the underlying asset $\{S_t\}_{t\in[0,T]}$ in the interval $[0, T]$. In this chapter, we consider four types of Asian call options with payoffs V_T at the time of maturity T for a given strike price $K > 0$:

1. **Geometric average with continuous sampling:**
$$V_T = \left(\exp\left(\frac{1}{T}\int_0^T \ln(S_t)\,dt\right) - K\right)^+.$$
2. **Geometric average with discrete sampling:**
$$V_T = \left(\left(\prod_{i=1}^n S_{t_i}\right)^{\frac{1}{n}} - K\right)^+,$$

where $0 < t_1 < t_2 < \cdots < t_n = T$.

3. **Arithmetic average with continuous sampling:**

$$V_T = \left(\frac{1}{T} \int_0^T S_t \, dt - K \right)^+$$

4. **Arithmetic average with discrete sampling:**

$$V_T = \left(\frac{1}{n} \sum_{i=1}^n S_{t_i} - K \right)^+ .$$

The payoffs of Asian put options can be defined analogously.

In the sequel, we derive the closed-form risk-neutral pricing formula for Asian options with geometric average, which is given by the ***Kemna–Vorst formula***. For Asian options with arithmetic average, closed-form pricing formulae are lacking in general. Instead, we shall characterize the price with both risk-neutral expectation and the Feynman–Kac formula, which can be solved using numerical methods such as Monte-Carlo simulations and finite difference schemes, respectively.

For simplicity, we shall consider a simple Black-Scholes model for the price of the underlying asset:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t.$$

The risk-free interest rate is given by $r \geq 0$.

2 Geometric Average with Continuous Sampling

We derive the closed-form pricing formula for Asian call options with geometric average, whose payoff is given by

$$V_T = \left(\exp\left(\frac{1}{T} \int_0^T \ln(S_t) \, dt \right) - K \right)^+ .$$

We first consider a probabilistic approach by exploiting the log-normal property of S. In the second approach, we characterize the price by introducing an additional state variable.

2.1 Risk-Neutral Approach

Under the risk-neutral probability measure $\widetilde{\mathbb{P}}$, the process $\widetilde{B}_t := B_t + \theta t$, $\theta := \frac{\mu - r}{\sigma}$, is a standard Brownian motion. The price of the risky asset is then governed by the SDE

$$dS_t = r S_t \, dt + \sigma S_t \, d\widetilde{B}_t,$$

whose solution is given explicitly by

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \widetilde{B}_t\right).$$

The risk-neutral price V_t of the call option at time t is given by

$$V_t = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\exp\left(\frac{1}{T}\int_0^T \ln(S_u)\,du\right) - K\right)^+ \Big|\mathcal{F}_t\right].$$

To compute the risk-neutral price, we examine the distribution of

$$\frac{1}{T}\int_0^T \ln(S_u)\,du = \frac{1}{T}\int_0^t \ln(S_u)\,du + \frac{1}{T}\int_t^T \ln(S_u)\,du.$$

The first term of the above is $\mathcal{F}_t$-measurable. Henceforth, we focus on the second term:

$$\frac{1}{T}\int_t^T \ln(S_u)\,du = \frac{1}{T}\left[(T-t)\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma\int_t^T \left(\widetilde{B}_u - \widetilde{B}_t\right)\,du\right].$$

Note that $\widetilde{B}_u - \widetilde{B}_t \sim \mathcal{N}(0, u-t)$ under $\widetilde{\mathbb{P}}$. Hence, the integral $I_{t,T} := \int_t^T (\widetilde{B}_u - \widetilde{B}_t)\,du$ is also Gaussian distributed. The mean of $I_{t,T}$ is given by

$$\widetilde{\mathbb{E}}[I_{t,T}|\mathcal{F}_t] = \widetilde{\mathbb{E}}[I_{t,T}] = \int_t^T \widetilde{\mathbb{E}}[\widetilde{B}_u - \widetilde{B}_t]\,du = 0$$

On the other hand, its variance can be computed by

$$\begin{aligned}
\widetilde{\mathrm{Var}}[I_{t,T}|\mathcal{F}_t] &= \widetilde{\mathrm{Var}}[I_{t,T}] \\
&= \widetilde{\mathbb{E}}\left[\left(\int_t^T (\widetilde{B}_u - \widetilde{B}_t)^2\,du\right)\right] \\
&= \widetilde{\mathbb{E}}\left[\int_t^T \int_t^T (\widetilde{B}_u - \widetilde{B}_t)(\widetilde{B}_v - \widetilde{B}_t)\,du\,dv\right] \\
&= \widetilde{\mathbb{E}}\left[\int_t^T \int_t^v (\widetilde{B}_u - \widetilde{B}_t)(\widetilde{B}_v - \widetilde{B}_t)\,du\,dv + \int_t^T \int_v^T (\widetilde{B}_u - \widetilde{B}_t)(\widetilde{B}_v - \widetilde{B}_t)\,du\,dv\right]
\end{aligned}$$

Note that for $t \le u \le v$, using independent increment of Brownian motions,

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(\widetilde{B}_u - \widetilde{B}_t)(\widetilde{B}_v - \widetilde{B}_t)\right] &= \widetilde{\mathbb{E}}\left[(\widetilde{B}_u - \widetilde{B}_t)[(\widetilde{B}_v - \widetilde{B}_u) + (\widetilde{B}_u - \widetilde{B}_t))\right] \\
&= \widetilde{\mathbb{E}}[\widetilde{B}_u - \widetilde{B}_t]\widetilde{\mathbb{E}}[\widetilde{B}_v - \widetilde{B}_u] + \widetilde{\mathbb{E}}\left[(\widetilde{B}_u - \widetilde{B}_t)^2\right] \\
&= u - t.
\end{aligned}$$

By symmetry, for $t \le v \le u$,

$$\widetilde{\mathbb{E}}\left[(\widetilde{B}_u - \widetilde{B}_t)(\widetilde{B}_v - \widetilde{B}_t)\right] = v - t.$$

Substituting these back to the expression of the variance, we obtain

$$\widetilde{\mathrm{Var}}[I_{t,T}|\mathcal{F}_t] = \int_t^T \int_t^v (u-t)\,du\,dv + \int_t^T \int_v^T (v-t)\,du\,dv = \frac{(T-t)^3}{3}.$$

Therefore, given $\mathcal{F}_t$,

$$\frac{1}{T}\int_0^T \ln(S_u)\,du\bigg|_{\mathcal{F}_t} \sim \mathcal{N}(m_t, v_t),$$

where

$$m_t = \frac{1}{T}\left[\int_0^t \ln(S_u)\,du + \left(\ln(S_t) + \frac{T-t}{2}\left(r - \frac{\sigma^2}{2}\right)\right)(T-t)\right],\ v_t = \frac{\sigma^2(T-t)^3}{3T^2}.$$

Using the distribution of $\frac{1}{T}\int_0^T \ln(S_u)\,du$, we can write

$$\frac{1}{T}\int_0^T \ln(S_u)\,du\bigg|_{\mathcal{F}_t} = m_t + \sqrt{v_t}Z,$$

where $Z \sim \mathcal{N}(0,1)$ under $\widetilde{\mathbb{E}}$. Hence,

$$\begin{aligned}
V_t &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\exp\left(\frac{1}{T}\int_0^T \ln(S_u)\,du\right) - K\right)^+ |\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(e^{m_t+\sqrt{v_t}Z} - K\right)\mathbb{1}_{\{e^{m_t+\sqrt{v_t}Z}>K\}}|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[(e^{m_t+\sqrt{v_t}Z} - K)\mathbb{1}_{\{Z>\frac{\ln K - m_t}{\sqrt{v_t}}\}}|\mathcal{F}_t\right] \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\frac{\ln K - m_t}{\sqrt{v_t}}}^{\infty} e^{m_t+\sqrt{v_t}z}e^{-\frac{z^2}{2}}\,dz - Ke^{-r(T-t)}N\left(\frac{m_t - \ln K}{\sqrt{v_t}}\right) \\
&= \frac{e^{m_t+\frac{v_t}{2}-r(T-t)}}{\sqrt{2\pi}}\int_{\frac{\ln K - m_t}{\sqrt{v_t}}}^{\infty} e^{-\frac{(z-\sqrt{v_t})^2}{2}}\,dz - Ke^{-r(T-t)}N\left(\frac{m_t - \ln K}{\sqrt{v_t}}\right) \\
&= \boxed{e^{m_t+\frac{v_t}{2}-r(T-t)}N\left(\frac{m_t - \ln K}{\sqrt{v_t}} + \sqrt{v_t}\right) - Ke^{-r(T-t)}N\left(\frac{m_t - \ln K}{\sqrt{v_t}}\right).} \qquad (1)
\end{aligned}$$

In particular, when $t = 0$,

$$m_0 = \ln S_0 + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T,\ v_0 = \frac{\sigma^2 T}{3}.$$

Hence, we arrive at the ***Kemna-Vorst formula:***

$$\boxed{V_0 = S_0 e^{-\frac{1}{2}\left(r+\frac{\sigma^2}{6}\right)T} N(\hat{d}_1) - K e^{-rT} N(\hat{d}_2),}$$

where

$$\hat{d}_1 := \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)T}{\sigma_G\sqrt{T}}, \ \hat{d}_2 := \hat{d}_1 - \sigma_G\sqrt{T}, \ \sigma_G := \frac{\sigma}{\sqrt{3}}.$$

Since a geometric Asian option is based on the *average* of the underlying asset, its effective volatility is reduced: in continuous time, $\sigma_G = \sigma/\sqrt{3}$. Averaging smooths out fluctuations, making the option less sensitive to short-term volatility and generally cheaper than a European option on the terminal spot.

2.2 PDE Approach

Due to the path-dependent nature, the payoff function of Asian options are not Markovian in the price of the risky asset S. Therefore, the (1-dimensional) Feynman–Kac formula is not applicable when computing the risk-neutral expectation. To address this, we introduce the running sum of the log-price as an additional state variable:

$$Y_t := \int_0^t \ln S_u \, du, \quad t > 0.$$

Using this, we can consider the pair (S, Y) as the solution of the following two-dimensional SDE under the risk-neutral measure:

$$\begin{cases} dS_t = rS_t\, dt + \sigma S_t\, d\widetilde{B}_t, \\ dY_t = \ln S_t\, dt. \end{cases} \tag{2}$$

Aligning with the multidimensional SDE introduced in the last chapter, we see that $m = 2$, $d = 1$, and

$$b_1(t, s, y) = rs, \ b_2(t, s, y) = \ln s, \ \sigma^{11}(t, s, y) = \sigma s, \ \sigma^{21}(t, s, y) = 0.$$

The payoff function of the geometric Asian option can now be expressed as $V_T = f(S_T, Y_T)$, where

$$f(s, y) = \left(e^{\frac{y}{T}} - K\right)^+$$

Define

$$V(t, s, y) := \widetilde{\mathbb{E}}^{t,s,y}\left[e^{-r(T-t)}V_T\right] = \widetilde{\mathbb{E}}^{t,s,y}\left[e^{-r(T-t)}f(S_T, Y_T)\right]. \tag{3}$$

By the Markov property of the two-dimensional SDEs (S, Y), we know that

$$V(t, S_t, Y_t) = \widetilde{\mathbb{E}}^{t,S_t,Y_t}\left[e^{-r(T-t)}f(S_T, Y_T)\right] = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}f(S_T, Y_T)|\mathcal{F}_t\right].$$

Hence, $V(t, S_t, Y_t)$ gives the risk-neutral price of the risk-neutral price of the option at time t given the state (S_t, Y_t). By the multidimensional Feynman–Kac formula, the function $V(t, s, y)$ is the solution of the following PDE:

$$\begin{cases} V_t + rsV_s + \dfrac{\sigma^2 s^2}{2} V_{ss} + (\ln s)V_y - rV = 0, & (t, s, y) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}, \\ V(T, s, y) = \left(e^{y/T} - K\right)^+, & (s, y) \in \mathbb{R}_+ \times \mathbb{R}. \end{cases} \tag{4}$$

Equation (4) is a two-dimensional partial differential equation in the spatial variables s and y. By introducing an appropriate change of variables, the equation can be reduced to a one-dimensional parabolic PDE. This dimension reduction not only clarifies the underlying structure of the problem, but also facilitates both the derivation of closed-form solutions and the implementation of numerical methods, such as finite-difference schemes.

To this end, we introduce the variables

$$\tau := T - t,\ z := y + \tau \ln s,\ u(\tau, z) := V(t, s, y).$$

Then, we have

$$V_t = -u_\tau - (\ln s)u_z,\ V_s = \frac{\tau}{s} u_z,\ V_y = u_z,\ V_{ss} = \left(\frac{\tau}{s}\right)^2 u_{zz} - \frac{\tau}{s^2} u_z. \tag{5}$$

Substituting (5) into (4), we have

$$\begin{aligned} 0 &= (-u_\tau - (\ln s)u_z) + r\tau u_z + \frac{\sigma^2}{2}\left(\tau^2 u_{zz} - \tau u_z\right) + (\ln s)u_z - ru \\ &= -u_\tau - \tau\left(r + \frac{\sigma^2}{2}\right) u_z + \frac{\sigma^2 \tau^2}{2} u_{zz} - ru. \end{aligned}$$

The boundary condition becomes

$$V(T, s, y) = u(0, z) = \left(e^{\frac{y + (T-T)\ln s}{T}} - K\right)^+ = \left(e^{z/T} - K\right)^+.$$

Therefore, we obtain the following one-dimensional PDE:

$$\begin{cases} u_\tau(\tau, z) = \dfrac{\sigma^2 \tau^2}{2} u_{zz}(\tau, z) + \tau\left(r - \dfrac{\sigma^2}{2}\right) u_z(\tau, z) - ru(\tau, z), & \tau \in [0, T], z \in \mathbb{R}, \\ u(0, z) = \left(e^{z/T} - K\right)^+, & z \in \mathbb{R}. \end{cases} \tag{6}$$

By applying additional transformations, equation (6) can be reduced to a standard heat equation, whose solution is expressed in terms of a Gaussian kernel and coincides with the Kemna–Vorst formula. We leave the details of this derivation as an exercise.

3 Geometric Average with Discrete Sampling

In this section, we derive the price of an Asian option with the payoff

$$V_T = \left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} - K\right)^+,$$

where $0 < t_1 < t_2 < \cdots < t_n = T$. We also let $t_0 := 0$.

The risk-neutral price V_t of the option at time $t < T$ is given by

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} - K\right)^+ \Big| \mathcal{F}_t\right].$$

Let $k \in \{0, 1, \ldots, n-1\}$ be such that $t_k \leq t < t_{k+1}$. Then $\mathcal{F}_{t_k} \subseteq \mathcal{F}_t \subseteq \mathcal{F}_{t_{k+1}}$. To compute the price, we determine the conditional distribution of the geometric average given $\mathcal{F}_t$.

Consider

$$\begin{aligned}
\ln\left[\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}\right] &= \frac{1}{n}\sum_{i=1}^{n}\ln(S_{t_i}) \\
&= \frac{1}{n}\left(\sum_{i=1}^{k}\ln(S_{t_i}) + \sum_{i=k+1}^{n}\ln(S_{t_i})\right) \\
&= \frac{1}{n}\sum_{i=1}^{k}\ln(S_{t_i}) + \frac{1}{n}\sum_{i=k+1}^{n}\left[\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(t_i - t) + \sigma(\widetilde{B}_{t_i} - \widetilde{B}_t)\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{k}\ln(S_{t_i}) + (n-k)\ln S_t + \sum_{i=k+1}^{n}\left(r - \frac{\sigma^2}{2}\right)(t_i - t)\right] + \frac{\sigma}{n}\sum_{i=k+1}^{n}(\widetilde{B}_{t_i} - \widetilde{B}_t).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{i=k+1}^{n}(\widetilde{B}_{t_i} - \widetilde{B}_t) &= (n-k)(\widetilde{B}_{t_{k+1}} - \widetilde{B}_t) + (n-k-1)(\widetilde{B}_{t_{k+2}} - \widetilde{B}_{t_{k+1}}) \\
&\quad + \cdots + (\widetilde{B}_{t_n} - \widetilde{B}_{t_{n-1}}) \\
&= (n-k)(\widetilde{B}_{t_{k+1}} - \widetilde{B}_t) + \sum_{j=1}^{n-k-1}(n-k-j)\left(\widetilde{B}_{t_{k+1+j}} - \widetilde{B}_{t_{k+j}}\right).
\end{aligned}$$

By the independence and Gaussianity of Brownian increments, we have

$$\sum_{i=k+1}^{n} (\widetilde{B}_{t_i} - \widetilde{B}_t)\Big|\mathcal{F}_t \sim \mathcal{N}\left(0,\ (n-k)^2(t_{k+1}-t) + \sum_{j=1}^{n-k-1}(n-k-j)^2(t_{k+1+j}-t_{k+j})\right).$$

Therefore,

$$\ln\left[\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}\right]\Big|\mathcal{F}_t \sim \mathcal{N}(m_t^{dis}, v_t^{dis}), \tag{7}$$

where

$$\begin{aligned} m_t^{dis} &= \frac{1}{n}\left[\sum_{i=1}^{k}\ln(S_{t_i}) + (n-k)\ln S_t + \sum_{i=k+1}^{n}\left(r - \frac{\sigma^2}{2}\right)(t_i - t)\right], \\ v_t^{dis} &= \frac{\sigma^2}{n^2}\left((n-k)^2(t_{k+1}-t) + \sum_{j=1}^{n-k-1}(n-k-j)^2(t_{k+1+j}-t_{k+j})\right). \end{aligned} \tag{8}$$

Using the distribution of the geometric average, we can proceed to compute the risk-neutral price of the option. Express

$$\ln\left[\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}\right]\Big|\mathcal{F}_t = m_t + \sqrt{v_t}Z,$$

where $Z \sim \mathcal{N}(0,1)$ under $\widetilde{\mathbb{E}}$. Then, following the same calculations as in the derivation of (1), we have

$$\begin{aligned} V_t &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(\left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}} - K\right)^+ \Big|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(e^{m_t^{dis}+\sqrt{v_t^{dis}}Z} - K\right)\mathbb{1}_{\{e^{m_t^{dis}+\sqrt{v_t^{dis}}Z}>K\}}\Big|\mathcal{F}_t\right] \\ &= \boxed{e^{m_t^{dis}+\frac{v_t^{dis}}{2}-r(T-t)}N\left(\frac{m_t^{dis}-\ln K}{\sqrt{v_t^{dis}}} + \sqrt{v_t^{dis}}\right) - Ke^{-r(T-t)}N\left(\frac{m_t^{dis}-\ln K}{\sqrt{v_t^{dis}}}\right)}. \end{aligned}$$

4 Arithmetic Average with Continuous Sampling

In this section, we consider the pricing of Asian options with arithmetic average and continuous sampling, whose payoff at maturity is given by

$$V_T = \left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+.$$

By the risk-neutral pricing principle, the price of the option at time $t \in [0, T]$ is

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+ \,\Big|\, \mathcal{F}_t\right].$$

In contrast to the case of geometric averages, where the lognormality of the asset price process leads to a closed-form pricing formula, the arithmetic average involves the time integral of a lognormal diffusion. The distribution of this integral does not admit a closed-form expression, and consequently no closed-form pricing formula is available for arithmetic Asian options in general.

To compute the option price, one may therefore resort to numerical methods, such as Monte Carlo simulation. Alternatively, as we show below, the price process can be characterized as the solution to a partial differential equation, which may be solved numerically using finite-difference or related PDE-based methods.

Similar to the PDE approach for geometric average Asian options, we introduce an additional process $\{Y_t\}_{t\in[0,T]}$, defined by

$$Y_t := \int_0^t S_u\,du.$$

Then, the pair (S, Y) is the solution of the following two-dimensional SDE under the risk-neutral measure:

$$\begin{cases} dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{B}_t, \\ dY_t = S_t\,dt. \end{cases} \tag{9}$$

The payoff function of the arithmetic Asian option can now be expressed as $V_T = f(S_T, Y_T)$, where

$$f(s, y) = \left(\frac{y}{T} - K\right)^+$$

Define

$$V(t, s, y) := \widetilde{\mathbb{E}}^{t,s,y}\left[e^{-r(T-t)}V_T\right] = \widetilde{\mathbb{E}}^{t,s,y}\left[e^{-r(T-t)}f(S_T, Y_T)\right]. \tag{10}$$

By the Markov property of the two-dimensional SDEs (S, Y), we know that

$$V(t, S_t, Y_t) = \widetilde{\mathbb{E}}^{t,S_t,Y_t}\left[e^{-r(T-t)}f(S_T, Y_T)\right] = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}f(S_T, Y_T)|\mathcal{F}_t\right].$$

Hence, $V(t, S_t, Y_t)$ gives the risk-neutral price of the risk-neutral price of the option at time t given the state (S_t, Y_t). By the multidimensional Feynman–Kac formula, the function $V(t, s, y)$ is the solution of the following PDE:

$$\begin{cases} V_t(t, s, y) + rsV_s(t, s, y) + sV_y(t, s, y) + \dfrac{\sigma^2 s^2}{2}V_{ss}(t, s, y) - rV(t, s, y) = 0, \\ \qquad\qquad (t, s, y) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ V(T, s, y) = \left(\dfrac{y}{T} - K\right)^+, \ (s, y) \in \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{11}$$

Remark 4.1. Although the accumulated process $Y_t = \int_0^t S_u\, du$ is nonnegative, it is convenient to extend the spatial domain of the pricing PDE (4) to allow $y \in \mathbb{R}$. From the probabilistic representation, starting from any $y \in \mathbb{R}$ we have $Y_T = y + \int_t^T S_u\, du \geq 0$ almost surely, so the terminal payoff is well defined and depends only on nonnegative values of Y_T. Consequently, one may solve the PDE on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$ and restrict the solution to $y \geq 0$ to recover the risk-neutral pricing formula.

4.1 Dimension Reduction

Although Equation (11) does not admit a closed-form solution, we can apply a transformation to reduce the equation to a one-dimensional PDE. To this end, we define the variable $z \in \mathbb{R}$ and the function $u(t, z)$ in a way that

$$z := \frac{y - KT}{s} \quad \text{and} \quad V(t, s, y) = su(t, z).$$

Then, we have

$$\begin{aligned} &V_t = su_t,\ V_s = u + su_z \cdot \left(-\frac{y - KT}{s^2}\right) = u - zu_z,\ V_y = u_z, \\ &V_{ss} = u_z \cdot \left(-\frac{y - KT}{s^2}\right) - u_z \cdot \left(-\frac{y - KT}{s^2}\right) - zu_{zz} \cdot \left(-\frac{y - KT}{s^2}\right) = \frac{z^2}{s} u_{zz}. \end{aligned} \tag{12}$$

Substituting (12) into (11), we have

$$0 = su_t + rs(u - zu_z) + su_z + \frac{\sigma^2 s^2}{2} \cdot \frac{z^2}{s} u_{zz} - rsu = s\left[u_t + (1 - rz)u_z + \frac{\sigma^2}{2} z^2 u_{zz}\right].$$

On the other hand, since $s \geq 0$, the terminal condition reads as

$$V(T, s, y) = \left(\frac{y - KT}{T}\right)^+ = s\left(\frac{1}{T} \cdot \frac{y - KT}{s}\right)^+ = \frac{s}{T} z^+,$$

whence $u(T, z) = z^+/T$.

Summarizing the above, the price of the Asian option with arithmetic average and continuous sampling is given by

$$V(t, s, y) = su(t, z),$$

where u is the solution of the following one-dimensional PDE:

$$\begin{cases} u_t(t, z) + (1 - rz)u_z(t, z) + \dfrac{\sigma^2}{2} z^2 u_{zz}(t, z) = 0, & t \in [0, T], z \in \mathbb{R}, \\ u(T, z) = \dfrac{z^+}{T}, & z \in \mathbb{R}. \end{cases} \tag{13}$$

4.2 Change of Numéaire

The change of numéraire offers a probabilistic derivation of a one-dimensional PDE for pricing arithmetic-average Asian options. By choosing the stock price as the numéraire, the payoff can be written as $V_T = S_T(Y_T)^+$ for a suitably defined normalized process Y. Under the stock-numéraire measure, Y is Markovian, leading to a one-dimensional PDE known as the Večer equation. This approach extends naturally to discrete sampling. In this subsection, we assume that $r > 0$.

4.2.1 Replicating portfolio

The first step of the approach is to identify a self-financing, deterministic, portfolio strategy $\gamma_t, t \in [0, T]$, along with an initial capital X_0, such that the associated portfolio value process $\{X_t\}_{t\in[0,T]}$ has a terminal value given by

$$X_T = \frac{1}{T}\int_0^T S_t\,dt - K. \tag{14}$$

Recall that the dynamics of X under the self-financing strategy γ is given by

$$dX_t = \gamma_t\,dS_t + r(X_t - \gamma_t S_t)\,dt = rX_t\,dt + \gamma_t(dS_t - rS_t\,dt). \tag{15}$$

The discounted portfolio process, $e^{-rt}X_t$, thus satisfies

$$\begin{aligned} d(e^{-rt}X_t) &= -re^{-rt}X_t\,dt + e^{-rt}\,dX_t \\ &= \gamma_t e^{-rt}(dS_t - rS_t\,dt) \\ &= \gamma_t\,d(e^{-rt}S_t) \\ &= d(e^{-rt}\gamma_t S_t) - e^{-rt}S_t\,d\gamma_t, \end{aligned}$$

where the last line follows from the product rule and the fact that γ_t is assumed to be deterministic (and so there is no cross-variation term). Integrating both sides with respect to t, we obtain

$$\begin{aligned} e^{-rt}X_t &= X_0 + \int_0^t d(e^{-ru}\gamma_u S_u) - \int_0^t e^{-ru}S_u\,d\gamma_u \\ &= X_0 + e^{-rt}\gamma_t S_t - \gamma_0 S_0 - \int_0^t e^{-ru}S_u\gamma_u'\,du. \end{aligned}$$

In particular, when $t = T$, we have

$$X_T = e^{rT}X_0 + \gamma_T S_T - e^{rT}\gamma_0 S_0 - \int_0^T e^{r(T-t)}S_t\gamma_t'\,dt. \tag{16}$$

To construct the term $1/T\int_0^T S_t\,dt$ and eliminate $\gamma_T S_T$ we choose γ_t such that

$$\gamma_t' = -\frac{e^{-r(T-t)}}{T},\ \ \gamma_T = 0.$$

Solving this ODE yields

$$\gamma_t = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right). \tag{17}$$

Substituting (17) into (16), we obtain

$$X_T = e^{rT}X_0 - \frac{e^{rT}S_0}{rT}(1-e^{-rT}) + \frac{1}{T}\int_0^T S_t\,dt = \frac{1}{T}\int_0^T S_t\,dt + e^{rT}\left(X_0 - \frac{1}{rT}(1-e^{-rT})S_0\right).$$

Hence, by choosing

$$X_0 := \frac{1}{rT}(1-e^{-rT})S_0 - e^{-rT}K, \tag{18}$$

we arrive at (14) as desired.

4.2.2 Change of measure

From the replicating portfolio, we know that the price of the Asian option can alternatively be written as

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\left[X_T^+|\mathcal{F}_t\right], \tag{19}$$

where X follows the SDE (15), with the portfolio strategy and initial capital are given by (17) and (18), respectively.

The conditional expectation (19) can in principle be computed using the Feynman–Kac formula. However, the dynamics of X in (15) depend on S, resulting in a two-dimensional problem. To simplify pricing, we change to the ***stock-numéraire measure*** $\widetilde{\mathbb{P}}^S$, under which the process $Y_t := \frac{X_t}{S_t}$ is a martingale. This reduces the PDE from two dimensions to one, eliminates the drift term (removing the first-order derivative), and allows the price to be expressed as a conditional expectation of Y_T^+, facilitating analytical or semi-analytical solutions via the Večer equation.

We first provide the definition of the stock-numéraire measure.

Definition 4.1 (Stock-numéraire measure) Let S_t be a strictly positive stock price process. The ***stock-numéraire measure*** $\widetilde{\mathbb{P}}^S$ is the probability measure under which the normalized value of any self-financing portfolio X_t relative to the stock,

$$Y_t := \frac{X_t}{S_t},$$

is a martingale. That is, for $0 \le t \le T$,

$$Y_t = \widetilde{\mathbb{E}}^S[Y_T \mid \mathcal{F}_t].$$

Remark 4.2. Under the usual risk-neutral measure $\widetilde{\mathbb{P}}$, the numéraire is the risk-free asset $F_t = e^{rt}$, so that the discounted stock price $e^{-rt}S_t$ (and $e^{-rt}X_t$) is a martingale. The stock-numéraire measure is analogous, but we use the stock S_t itself as the numéraire.

We proceed to construct the measure $\widetilde{\mathbb{P}}^S$. To this end, we derive the dynamics of Y_t. Using Itô's lemma, we have

$$\begin{aligned} dS_t^{-1} &= -S_t^{-2}\,dS_t - \frac{1}{2}(-2)S_t^{-3}\,d\langle S\rangle_t \\ &= -S_t^{-1}(r\,dt + \sigma\,d\widetilde{B}_t) + \sigma^2 S_t^{-1}\,dt \\ &= S_t^{-1}\left[(\sigma^2 - r)\,dt - \sigma\,d\widetilde{B}_t\right]. \end{aligned}$$

On the other hand, from (15), we know that

$$dX_t = rX_t\,dt + \gamma_t(dS_t - rS_t\,dt) = rX_t\,dt + \sigma\gamma_t S_t\,d\widetilde{B}_t.$$

Hence, using the product rule, we have

$$\begin{aligned} dY_t = d(S_t^{-1}X_t) &= X_t\,dS_t^{-1} + S_t^{-1}\,dX_t + d\langle X, S^{-1}\rangle_t \\ &= X_t S_t^{-1}\left[(\sigma^2 - r)\,dt - \sigma\,d\widetilde{B}_t\right] + S_t^{-1}\left[rX_t\,dt + \sigma\gamma_t S_t\,d\widetilde{B}_t\right] - \sigma^2\gamma_t\,dt \\ &= \sigma^2 Y_t\,dt - \sigma Y_t\,d\widetilde{B}_t + \sigma\gamma_t\,d\widetilde{B}_t - \sigma^2\gamma_t\,dt \\ &= \sigma(\gamma_t - Y_t)(d\widetilde{B}_t - \sigma\,dt). \end{aligned} \tag{20}$$

Motivated by (20), we invoke Girsanov's theorem to define $\widetilde{P}^S$, under which Y would become a martingale. To this end, define the process $\{\widetilde{B}_t^S\}_{t\in[0,T]}$ by

$$\widetilde{B}_t^S := \widetilde{B}_t - \sigma t.$$

Then, the dynamics of Y can be written as

$$dY_t = \sigma(\gamma_t - Y_t)d\widetilde{B}_t^S.$$

Define also the density process $\{Z_t^S\}_{t\in[0,T]}$ by

$$Z_t^S := \exp\left(\sigma\widetilde{B}_t - \frac{1}{2}\sigma^2 t\right),$$

i.e.,

$$Z_t^S = \frac{e^{-rt}S_t}{S_0}. \tag{21}$$

It is easy to check that $\widetilde{\mathbb{E}}[Z_T^S] = 1$, and Therefore, by Girsanov's theorem, we can define a probability measure $\widetilde{\mathbb{P}}^S$ by

$$\widetilde{\mathbb{P}}^S(A) := \widetilde{\mathbb{E}}[\mathbb{1}_A Z_T^S],\ A \in \mathcal{F}_T,$$

such that $\{\widetilde{B}_t^S\}_{t\in[0,T]}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}^S$. In addition, Y is a $\widetilde{\mathbb{P}}^S$-martingale.

4.2.3 The Večer equation

We now compute the price of the Asian option using the probability measure $\widetilde{\mathbb{P}}^S$. Using the risk-neutral pricing formula and (21), we have

$$\begin{aligned}
V_t &= e^{-r(T-t)}\widetilde{\mathbb{E}}[X_T^+|\mathcal{F}_t] \\
&= e^{rt}\widetilde{\mathbb{E}}\left[e^{-rT}S_T\left(\frac{X_T}{S_T}\right)^+\Bigg|\mathcal{F}_t\right] \\
&= S_t\widetilde{\mathbb{E}}\left[\frac{e^{-rT}S_T/S_0}{e^{-rt}S_t/S_0}Y_T^+|\mathcal{F}_t\right] \\
&= \frac{S_t}{Z_t^S}\widetilde{\mathbb{E}}\left[Z_TY_T^+|\mathcal{F}_t\right] \\
&= S_t\widetilde{\mathbb{E}}^S[Y_T^+|\mathcal{F}_t],
\end{aligned}$$

where $\widetilde{\mathbb{E}}^S$ denotes the expectation taking with respect to $\widetilde{\mathbb{P}}^S$, and the last line follows from the Bayes theorem for conditional expectations.

Let $V(t,s,y)$ be the price of the Asian option at time t with $S_t = s$ and $Y_t = y$. Then,

$$V(t,S_t,Y_t) = S_t\widetilde{\mathbb{E}}^S[Y_T^+|\mathcal{F}_t].$$

Using the Markov property of the solution of the SDE Y in (20), there exists $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that

$$g(t,Y_t) = \widetilde{\mathbb{E}}^S[Y_T^+|\mathcal{F}_t].$$

By (20) and the Feynman–Kac formula, we know that g satisfies the following ***Večer equation***:

$$\begin{cases} g_t(t,y) + \frac{1}{2}\sigma^2(\gamma_t - y)^2 g_{yy}(t,y) = 0, \quad t \in [0,T],\ y \in \mathbb{R}; \\ g(T,y) = y^+,\ y \in \mathbb{R}. \end{cases} \tag{22}$$

Summarizing the above, we see that the price of the Asian option, $V(t,S_t,Y_t)$, is given by

$$V(t,S_t,Y_t) = S_t g(t,Y_t),$$

where g is the solution of the PDE (22).

Remark 4.3. The one-dimensional PDE (13) differs in form from the Ve***v***cer equation obtained via the change-of-numéraire approach. This discrepancy is due to the choice of state variable and numéraire. Nevertheless, the two formulations are equivalent and yield the same option price.

5 Arithmetic Average with Discrete Sampling

The change of numéaire approach can also be used to price an Asian option with arithmetic average and discrete sampling, whose payoff is given by

$$V_T = \left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ .$$

In that case, we can construct a portfolio with $\{X_t^{dis}\}_{t\in[0,T]}$ with an initial capital X_0^{dis} and self-financing strategy γ_t^{dis}, $t \in [0, T]$, respectively given by

$$\begin{cases} X_0^{dis} = e^{-rT} \left(\gamma_0 e^{rT} - \frac{1}{n} \right) S_0 - e^{-rT} K, \\ \gamma_t^{dis} = \frac{1}{n} \sum_{i=k}^{n} e^{-r(T-t_i)}, \ t \in (t_{k-1}, t_k], \ k = 1, \dots, n, \\ \gamma_0^{dis} = \frac{1}{n} \sum_{i=0}^{n} e^{-r(T-t_i)}. \end{cases}$$

Then, one can show that

$$X_T^{dis} = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K.$$

Indeed, using

$$d(e^{-rt} X_t) = \gamma_t d(e^{-rt} S_t)$$

and the fact that $\gamma_t = \gamma_{t_k}$ for $t \in (t_{k-1}, t_k]$, integrating both sides yields

$$\begin{aligned} e^{-rT} X_T - X_0 &= \int_0^T \gamma_t \, d(e^{-rt} S_t) \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \gamma_t \, d(e^{-rt} S_t) \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \gamma_{t_k} \, d(e^{-rt} S_t) \\ &= \sum_{k=1}^{n} \gamma_{t_k} \left(e^{-rt_k} S_{t_k} - e^{-rt_{k-1}} S_{t_{k-1}} \right). \end{aligned}$$

Using the form of γ_{t_k} and changing the order of summation, we further have

$$e^{-rT} X_T - X_0 = \frac{1}{n} \sum_{k=1}^{n} \sum_{i=k}^{n} e^{-r(T-t_i)} \left(e^{-rt_k} S_{t_k} - e^{-rt_{k-1}} S_{t_{k-1}} \right)$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{i} e^{-r(T-t_i)}\left(e^{-rt_k}S_{t_k} - e^{-rt_{k-1}}S_{t_{k-1}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)}\left(e^{-rt_i}S_{t_i} - S_0\right) \\
&= \frac{e^{-rT}}{n}\sum_{i=1}^{n} S_{t_i} - \frac{S_0}{n}\sum_{i=1}^{n} e^{-r(T-t_i)} \\
&= \frac{e^{-rT}}{n}\sum_{i=1}^{n} S_{t_i} - S_0\left(\gamma_0 - \frac{1}{n}e^{-rT}\right)
\end{aligned}
$$

Therefore,

$$
X_T = e^{rT}X_0 + \frac{1}{n}\sum_{i=1}^{n} S_{t_i} - S_0\left(\gamma_0 e^{rT} - \frac{1}{n}\right).
$$

Choosing X_0 as given above yields the desired X_T.

The remainder of the derivations is the same as the case with continuous sampling, that is, the price of the Asian option is given by $V^{dis}(t, S_t, Y_t^{dis}) = S_t g^{dis}(t, Y_t^{dis})$, where $Y_t^{dis} = X_t^{dis}/S_t$, and g^{dis} is the solution of the following PDE:

$$
\begin{cases}
g_t^{dis}(t,y) + \frac{1}{2}\sigma^2(\gamma_t^{dis} - y)^2 g_{yy}^{dis}(t,y) = 0, \quad t \in [0,T],\ y \in \mathbb{R}; \\
g^{dis}(T,y) = y^+,\ y \in \mathbb{R}.
\end{cases}
$$

Comparing with (22), the difference of the PDE emerges from the definition of γ and γ^{dis}.

Further Readings

1. The change of numéaire approach for pricing Asian options with arithmetic average for continuous sampling when $r = 0$.