MATH5636 Stochastic Calculus for Finance II Spring 2026

Chapter 4: More About Brownian Motions

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Preview

In MATH 5635, we introduced a number of properties of Brownian motions, including Markov, martingale, and time-scaling. This chapter covers additional distributional properties of Brownian motions, including the first passage time, reflection principle, and their maximal processes. These distributional properties pave ways to pricing of exotic options under the Black-Scholes models, which depend on the pathwise behavior of the underlying stock price process.

Key topics in this chapter:

1. Stopping times;
2. First passage time and reflection principle;
3. Maximum of Brownian motions;
4. Maximum of Brownian motions with drift.

1 Reflection Principle and First Passage Time

Throughout this chapter, we will let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Let $m \in \mathbb{R}$. This section is concerned with the distribution of the first time τ_m the Brownian motion hits the level m. Mathematically, it is given by

$$\tau_m := \inf\{t \geq 0 : B_t = m\}. \tag{1}$$

We call τ_m the ***first passage time*** to the level m, which is an $\mathcal{F}$-measurable random variable.

1.1 Stopping Time

The first passage time is an example of a ***stopping time***, which is defined below:

Definition 1.1 (Stopping time) A non-negative random variable τ is said to be a ***stopping time*** in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ if, for any $t \geq 0$, the event $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable.

Remark 1.1. By definition, any $t \geq 0$ is a stopping time since $\{s \leq t\}$ is either Ω or $\varnothing$, for any $s, t \geq 0$.

Intutively, if τ is a stopping time, then at any time $t \geq 0$, one can determine whether the stopping event has already occurred using only the information available up to t. The following shows that the first passage time is indeed a stopping time.

Theorem 1.2 Let $(X_t)_{t\geq 0}$ be an adapted stochastic process with continuous sample paths, and let $a \in \mathbb{R}$ be a constant. Define the first passage time

$$\tau := \inf\{t \geq 0 : X_t = a\}.$$

Then τ is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Proof. Fix $t \geq 0$. Without loss of generality, suppose that $a > X_0$. By definition of τ,

$$\{\tau \leq t\} = \left\{ \sup_{0\leq s\leq t} X_s \geq a \right\}.$$

Since X has continuous sample paths, one can show that $\sup_{0\leq s\leq t} X_s$ is $\mathcal{F}_t$-measurable. Hence,

$$\{\tau \leq t\} = \left\{ \sup_{0\leq s\leq t} X_s \geq a \right\} \in \mathcal{F}_t.$$

If $X_t < a$, we can instead consider the process $Y_t = -X_t$ and repeat the above argument. Finally, if $X_t = a$, we have $\tau = 0$. The proof is thus complete. $\square$

We shall prove a specific case of the ***optional sampling theorem*** in Theorem 1.4, which gives distributional properties of the random variable X_τ. We first show the following discrete-time optional sampling in the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}_0}, \mathbb{P})$. Note that a random time τ in this filtered probability space is said to be a stopping time if $\{\tau \leq n\} \in \mathcal{F}_n$ for any $n \in \mathbb{N}_0$.

Lemma 1.3 Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a martingale in the space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}_0}, \mathbb{P})$, and τ be a stopping time. Then, $\{X_{\tau\wedge n}\}_{n\in\mathbb{N}_0}$ is a martingale. In addition, if τ is bounded, i.e., there exists $N < \infty$ such that $\mathbb{P}(\tau \leq N) = 1$, then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Proof. We first verify that $X_{\tau\wedge n}$ is a martingale.

1. Adaptedness. Since τ is a stopping time, the event $\{\tau \leq n\}$ is $\mathcal{F}_n$-measurable. We can write
$$X_{n\wedge\tau} = \sum_{k=0}^{n} X_k \mathbb{1}_{\{\tau=k\}} + X_n \mathbb{1}_{\{\tau>n\}}.$$
Each X_k is $\mathcal{F}_k \subset \mathcal{F}_n$-measurable, and each indicator $\mathbf{1}_{\{\tau=k\}}$ and $\mathbf{1}_{\{\tau>n\}}$ is $\mathcal{F}_n$-measurable. Hence $X_{n\wedge\tau}$ is $\mathcal{F}_n$-measurable.

2. Integrability. Since $n \wedge \tau \leq n$,
$$|X_{n\wedge\tau}| \leq \max_{0\leq k\leq n} |X_k|,$$
which is integrable since each X_k, $k \leq n$, is integrable.

3. Fix $n \geq 0$. We shall show that
$$\mathbb{E}[X_{(n+1)\wedge\tau} \mid \mathcal{F}_n] = X_{n\wedge\tau}.$$
Note that
$$X_{(n+1)\wedge\tau} = X_{(n+1)\wedge\tau}\mathbb{1}_{\{\tau\leq n\}} + X_{(n+1)\wedge\tau}\mathbb{1}_{\{\tau>n\}} = X_{n\wedge\tau}\mathbb{1}_{\{\tau\leq n\}} + X_{n+1}\mathbb{1}_{\{\tau>n\}}.$$
Hence,
$$\begin{aligned}
\mathbb{E}[X_{(n+1)\wedge\tau} \mid \mathcal{F}_n] &= \mathbb{E}\left[X_{n\wedge\tau}\mathbb{1}_{\{\tau\leq n\}} + X_{n+1}\mathbb{1}_{\{\tau>n\}} \mid \mathcal{F}_n\right] \\
&= X_{n\wedge\tau}\mathbb{1}_{\{\tau\leq n\}} + \mathbb{1}_{\{\tau>n\}}\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \\
&= X_{n\wedge\tau}\mathbb{1}_{\{\tau\leq n\}} + X_n\mathbb{1}_{\{\tau>n\}} = X_{n\wedge\tau},
\end{aligned}$$
where the second line follows from the fact that $\{\tau > n\} \in \mathcal{F}_n$, and the last line follows from the martingale property of X_n.

For the second statement, note that $X_{N\wedge\tau} = X_\tau$, since $\tau \leq N$ a.s. Hence, by the martingale property of the stopping process, we have $\mathbb{E}[X_0] = \mathbb{E}[X_{N\wedge\tau}] = \mathbb{E}[X_\tau]$.

□

Next, we prove the continuous version of Lemma 1.3.

Theorem 1.4 Let $\{X_t\}_{t\geq0}$ be a martingale in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq0}, \mathbb{P})$ with right-continuous sample paths, and let τ be a bounded stopping time, i.e., there exists $T < \infty$ such that $\mathbb{P}(\tau \leq T) = 1$. Then, $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Remark 1.5. With more effort, one can further show that the ***stopped process*** $\{X_{t\wedge\tau}\}_{t\geq0}$ is a martingale for any stopping time (not necessarily bounded) τ. This result is known as the ***optional sampling theorem***.

Proof. For each integer $n \geq 1$, we discretize the interval $[0, T]$ using a dyadic partition. Let $t_k = \frac{kT}{2^n}$ for $k = 0, 1, \ldots, 2^n$. Define the discrete stopping time τ_n by rounding τ up to the nearest grid point:

$$\tau_n := \sum_{k=1}^{2^n} t_k \mathbb{1}_{\{t_{k-1} < \tau \leq t_k\}}.$$

This definition ensures that $\tau_n \geq \tau$ and $\tau_n \downarrow \tau$ a.s. as $n \to \infty$.

Define the discrete filtration $\{\mathcal{G}_k^{(n)}\}_{k=0}^{2^n}$ by $\mathcal{G}_k^{(n)} := \mathcal{F}_{t_k}$. Note that τ_n is a stopping time with respect to $\{\mathcal{G}_k^{(n)}\}_{k=0}^{2^n}$, since for any grid point t_k, $\{\tau_n \leq t_k\} = \{\tau \leq t_k\} \in \mathcal{F}_{t_k} = \mathcal{G}_k^{(n)}$, since τ is a stopping time for the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Now, define $Y_k^{(n)} := X_{t_k}$. Since $\{X_t; \mathcal{F}_t\}_{t \geq 0}$ is a martingale, $\{Y_k^{(n)}; \mathcal{G}_k^{(n)}\}_{k=0}^{2^n}$ is a discrete-time martingale. By Lemma 1.3,

$$\mathbb{E}[X_{\tau_n}] = \mathbb{E}[Y_{\tau_n \cdot 2^n / T}^{(n)}] = \mathbb{E}[Y_0^{(n)}] = \mathbb{E}[X_0].$$

Finally, since $\tau_n \downarrow \tau$ and the sample paths of $\{X_t\}_{t \geq 0}$ are right-continuous by assumption, we have $X_{\tau_n} \to X_\tau$ a.s. as $n \to \infty$. Under a uniform integrability argument, we have

$$\mathbb{E}[X_\tau] = \lim_{n \to \infty} \mathbb{E}[X_{\tau_n}] = \mathbb{E}[X_0].$$

□

Using Theorem 1.4, we shall show the finiteness of the passage time τ_m defined in (1) and its finiteness.

Theorem 1.6 For any $m \in \mathbb{R}$, the first passage time of the Brownian motion to the level m satisfies $\mathbb{P}(\tau_m < \infty) = 1$. In addition, the MGF of τ_m is given by, for any $\alpha > 0$,

$$\mathbb{E}[e^{-\alpha \tau_m}] = e^{-|m|\sqrt{2\alpha}}. \tag{2}$$

Proof. We first show the finiteness of τ_m. To this end, for any $\sigma \geq 0$, consider the exponential martingale process $\{Z_t\}_{t \geq 0}$:

$$Z_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right).$$

The fact that Z_t is a martingale is a consequence of Girsanov's theorem. Since $t \wedge \tau_m$ is a bounded stopping time, by Theorem 1.4, we have

$$1 = \mathbb{E}[Z_0] = \mathbb{E}[Z_{t \wedge \tau_m}] = \mathbb{E}\left[\exp\left(\sigma B_{t \wedge \tau_m} - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\right)\right]. \tag{3}$$

Note that if $m = 0$, $\tau_0 = 0$ by the fact that $B_0 = 0$, and the result is trivial. Without loss of generality, we assume that $m > 0$; the case for $m < 0$ can be shown similarly. Since

$t \wedge \tau_m \leq \tau_m$, and the process B has not yet reached the level $m > 0$ before $t \wedge \tau_m$. Given that $B_0 = 0$, we must have $B_{t\wedge\tau_m} \leq m$. Hence,

$$Z_{t\wedge\tau_m} \leq e^{\sigma m} e^{-\frac{1}{2}\sigma^2(t\wedge\tau_m)} \leq e^{\sigma m}. \tag{4}$$

Decompose $Z_{t\wedge\tau_m}$ as

$$\begin{aligned} Z_{t\wedge\tau_m} &= Z_{t\wedge\tau_m} \mathbb{1}_{\{\tau_m<\infty\}} + Z_{t\wedge\tau_m} \mathbb{1}_{\{\tau_m=\infty\}} \\ &= \exp\left(\sigma B_{t\wedge\tau_m} - \frac{1}{2}\sigma^2(t\wedge\tau_m)\right) \mathbb{1}_{\{\tau_m<\infty\}} + \exp\left(\sigma B_{t\wedge\tau_m} - \frac{1}{2}\sigma^2 t\right) \mathbb{1}_{\{\tau_m=\infty\}}. \end{aligned}$$

Using the fact that $B_{t\wedge\tau_m} \leq m$, by passing to the limit $t \to \infty$, we have

$$\lim_{t\to\infty} Z_{t\wedge\tau_m} = e^{\sigma B_{\tau_m} - \frac{1}{2}\sigma^2\tau_m} \mathbb{1}_{\{\tau_m<\infty\}} = e^{\sigma m - \frac{1}{2}\sigma^2\tau_m} \mathbb{1}_{\{\tau_m<\infty\}}.$$

By (4), we can apply the dominated convergent theorem to (3), which yields

$$1 = \lim_{t\to\infty} \mathbb{E}[Z_{t\wedge\tau_m}] = \mathbb{E}\left[\lim_{t\to\infty} Z_{t\wedge\tau_m}\right] = e^{\sigma m}\mathbb{E}\left[e^{-\frac{1}{2}\sigma^2\tau_m} \mathbb{1}_{\{\tau_m<\infty\}}\right].$$

In particular, by choosing $\sigma = 0$, we obtain

$$1 = \mathbb{E}[\mathbb{1}_{\{\tau_m<\infty\}}] = \mathbb{P}(\tau_m < \infty).$$

Using the finiteness, we in turn have $\mathbb{1}_{\{\tau_m<\infty\}} = 1$ a.s., whence

$$1 = e^{\sigma m}\mathbb{E}\left[e^{-\frac{1}{2}\sigma^2\tau_m} \mathbb{1}_{\{\tau_m<\infty\}}\right] = e^{\sigma m}\mathbb{E}\left[e^{-\frac{1}{2}\sigma^2\tau_m}\right].$$

Now, the MGF can be derived by substituting $\sigma = \sqrt{2\alpha}$.

□

1.2 Reflection Principle and the Strong Markov Property

In this subsection, we further derive the density function for the first passage time τ_m for $m \neq 0$. To this end, we introduce the ***reflection principle***. Below we give a heuristic proof of the principle.

For any $m > 0$, we have

$$\mathbb{P}(\tau_m \leq t) = \mathbb{P}(\tau_m \leq t,\ B_t > m) + \mathbb{P}(\tau_m \leq t,\ B_t \leq m).$$

Note that if $B_t > m$, we must have $\tau_m \leq t$. Hence, $\{B_t > m\} \subseteq \{\tau_m \leq t\}$ and $\mathbb{P}(\tau_m \leq t,\ B_t > m) = \mathbb{P}(B_t > m)$.

On the other hand, if $\tau_m \le t$ and $B_t \le m$, then the path B_t must have reached the level m before t, and traveled to point $x \le m$ from the time interval $[\tau_m, t]$. By the symmetric property of Brownian motions, the probability of traveling from m to x is the same as the probability of its reflected path traveling from m to $2m - x$, since for each case, the path has traveled at a distance of $m - x$; see Figure 1. Since B_t and its reflected path share the same distribution, the above argument indicates that, for any $m > 0$ and $x \le m$,

$$\mathbb{P}(\tau_m \le t,\ B_t \le x) = \mathbb{P}(B_t \ge 2m - x).$$

In particular, by taking $x = m$, we have

$$\mathbb{P}(\tau_m \le t,\ B_t \le m) = \mathbb{P}(B_t \ge m) = \mathbb{P}(B_t > m).$$

Summarizing the above, we have

$$\begin{aligned}\mathbb{P}(\tau_m \le t) &= \mathbb{P}(\tau_m \le t,\ B_t > m) + \mathbb{P}(\tau_m \le t,\ B_t \le m)\\ &= \mathbb{P}(B_t > m) + \mathbb{P}(B_t > m) = 2\mathbb{P}(B_t > m).\end{aligned}$$

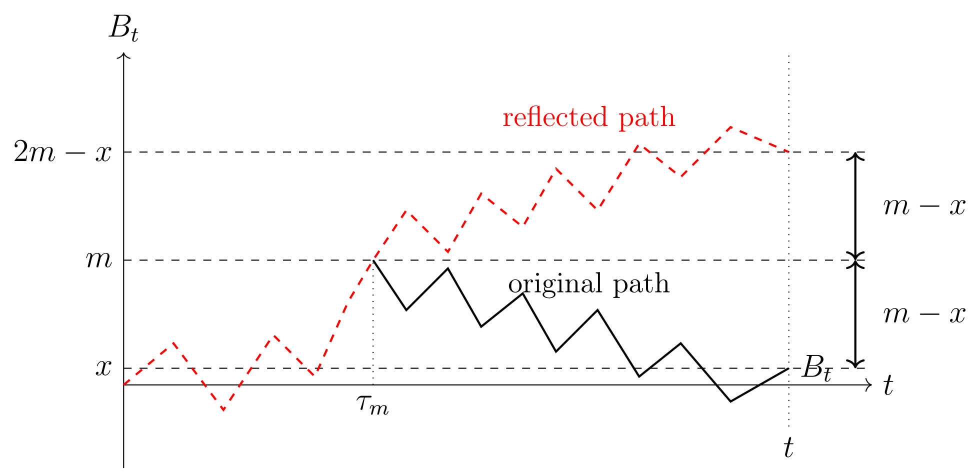

Figure 1: Illustration of the reflection principle for Brownian motion

Theorem 1.7 (Reflection principle for Brownian motions) For any $m > 0$ and $x \le m$,

$$\mathbb{P}\left(\tau_m \le t,\ B_t \le x\right) = \mathbb{P}\left(B_t \ge 2m - x\right).$$

In addition,

$$\mathbb{P}(\tau_m \le t) = 2\mathbb{P}(B_t > m).$$

Indeed, to prove the reflection principle rigorously, one implicitly invokes the ***strong Markov property*** of Brownian motion. In particular, when conditioning on the event

$\{\tau_m \le t\}$ and analyzing the behavior of the process after the hitting time τ_m, the property ensures that the post-τ_m increments are independent of the pre-τ_m history and have the same law as a fresh Brownian motion. This "restart" property guarantees that the reflected path exhibits the required distributional symmetry. Since the property holds for any stopping time (instead of just deterministic time t), it is stronger than the usual Markov property. Below, we state the definition of the strong Markov property, without providing a proof that Brownian motion satisfies it.

Definition 1.2 (Strong Markov Property) Let $\{X_t\}_{t\ge 0}$ be an adapted process in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$. For a stopping time τ, define *stopped sigma-algebra* $\mathcal{F}_\tau$[a] by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0\}.$$

We say that X satisfies the ***strong Markov property*** if for any stopping time τ and $t \ge 0$,

$$\mathbb{P}(X_{\tau+t} \in A | \mathcal{F}_\tau) = \mathbb{P}(X_{\tau+t} \in A | \sigma(X_\tau)).$$

[a] One can show that $\mathcal{F}_\tau$ is indeed a sub-σ-algebra of $\mathcal{F}$

Theorem 1.8 The Brownian motion is a strong Markov process.

1.3 Distribution Function of First Passage Time

Using the reflection principle, we can derive the density function of τ_m. Since $B_t \sim \mathcal{N}(0, t)$, using the reflection principle, the CDF of τ_m, $m > 0$, is given by

$$\mathbb{P}(\tau_m \le t) = 2\mathbb{P}(B_t > m) = 2\left(1 - N\left(\frac{m}{\sqrt{t}}\right)\right) = 2\int_{\frac{m}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx,\ t > 0.$$

By a direct differentiation, the density function of τ_m, $m > 0$, is given by

$$f_{\tau_m}(t) = \frac{m}{t}\frac{1}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}},\ t > 0. \tag{5}$$

If $m < 0$, by the symmetry of Brownian motions, we have $\tau_m \overset{d}{=} \tau_{|m|}$. Therefore, for any $m \ne 0$, the CDF and PDF of τ_m is given by

$$\mathbb{P}(\tau_m \le t) = 2\int_{\frac{|m|}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx, t > 0,$$
$$f_{\tau_m}(t) = \frac{|m|}{t}\frac{1}{\sqrt{2\pi t}} e^{-\frac{|m|^2}{2t}},\ t > 0.$$

One can use the PDF of τ_m to verify that its MGF is indeed given by (2).

2 Maximum of Brownian Motions

We define the ***running maximum*** $\{M_t\}_{t\geq 0}$ of the Brownian motion B, which is the highest value of the process up to the current time:

$$M_t := \max_{0\leq s\leq t} B_s,\ t\geq 0. \tag{6}$$

We shall derive the joint distribution of (M_t, B_t) using the reflection principle. Note that the pair takes values in $\{(m,x) : x\leq m,\ m\geq 0\}$, since $M_t\geq B_0=0$, and $B_t\leq M_t$ for any $t\geq 0$.

Recall that for any $m>0$ and $x\leq m$, the reflection principle indicates

$$\mathbb{P}(\tau_m\leq t,\ B_t\leq x)=\mathbb{P}(B_t\geq 2m-x).$$

For any $m>0$, the event $\{\tau_m\leq t\}$ means that the Brownian motion B must have reached the level m on or before time t, which is possible if and only if $M_t\geq m$. In other words, $\{\tau_m\leq t\}=\{M_t\geq m\}$, whence

$$\mathbb{P}(M_t\geq m,\ B_t\leq x)=\mathbb{P}(B_t\geq 2m-x),\ x\leq m,\ m>0. \tag{7}$$

Using (7), we can derive the joint PDF of (M_t, B_t).

Theorem 2.1 For any $t>0$, the joint density function of (M_t, B_t) is given by

$$f_{M_t,B_t}(m,x)=\frac{2(2m-x)}{t\sqrt{2\pi t}}e^{-\frac{(2m-x)^2}{2t}},\ x\leq m,\ m>0. \tag{8}$$

Proof. By (7), we have

$$\begin{aligned}\int_{2m-x}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}\,dz &= \mathbb{P}(B_t\geq 2m-x)\\ &= \mathbb{P}(M_t\geq m,\ B_t\leq x)\\ &= \int_m^{\infty}\int_{-\infty}^{x} f_{M_t,B_t}(w,z)\,dz\,dw.\end{aligned}$$

Differentiating both sides of the above equation with respect to m yields

$$-\frac{2}{\sqrt{2\pi t}}e^{-\frac{(2m-x)^2}{2t}}=-\int_{-\infty}^{x} f_{M_t,B_t}(m,z)\,dz.$$

By further differentiating both sides with respect to x, we obtain

$$f_{M_t,B_t}(x,z)=\frac{2(2m-x)}{t\sqrt{2\pi t}}e^{-\frac{(2m-x)^2}{2t}}$$

as desired. $\square$

Using (8), we can derive the unconditional CDF for the running maximum M_t. We shall defer the formula in the next section when we discuss the distributions of more general Brownian motion with drift and its running maximum.

3 Maximum of Brownian Motions with Drift

In this section, we consider the distributions of the Brownian motion with drift $\alpha \in \mathbb{R}$, $\{B_t^\alpha\}_{t\geq 0}$, and it running maximum, $\{M_t^\alpha\}_{t\geq 0}$, which are respectively defined by

$$B_t^\alpha = \alpha t + B_t,\ M_t^\alpha := \max_{0\leq s\leq t} B_s^\alpha,\ t \geq 0. \tag{9}$$

As in the case without drift, the pair (M_t^α, B_t^α) takes values in the set $\{(m,x) : x \leq m,\ m \geq 0\}$.

We derive the joint PDF of (M_t^α, B_t^α) by the following approach. First, by a change of measure, we can relate the distribution of (M_t^α, B_t^α) to that of the pair without drift, (M_t, B_t). Then, using the PDF of the drift-less pair (M_t, B_t) in (8), we can derive the PDF of (M_t^α, B_t^α) by multiplying formula (8) with an appropriate exponential factor, which corresponds to the density process in Girsanov's theorem.

Theorem 3.1 For any $t > 0$, the joint density function of (M_t^α, B_t^α) is given by

$$f_{M_t^\alpha, B_t^\alpha}(m,x) = \frac{2(2m-x)}{t\sqrt{2\pi t}} e^{\alpha x - \frac{1}{2}\alpha^2 t} e^{-\frac{(2m-x)^2}{2t}},\ x \leq m,\ m > 0. \tag{10}$$

Proof. For any $0 \leq s \leq t$, define the exponential martingale

$$Z_s := \exp\left(-\alpha B_s - \frac{1}{2}\alpha^2 s\right) = \exp\left(-\alpha B_s^\alpha + \frac{1}{2}\alpha^2 s\right),\ 0 \leq s \leq t.$$

By Girsanov's theorem, we can define a new probability measure $\widetilde{P}$ by

$$\widetilde{\mathbb{P}}(A) = \widetilde{\mathbb{E}}[\mathbb{1}_A Z_t],\ A \in \mathcal{F}_t,$$

such that $\{B_s^\alpha\}_{0\leq s\leq t}$ is a standard, driftless Brownian motion under $\widetilde{\mathbb{P}}$, and $\{M_s^\alpha\}_{0\leq s\leq t}$ is the running maximum of that standard Brownian motion. Under $\widetilde{\mathbb{P}}$, the joint PDF of (M_t^α, B_t^α) is given by $f_{M_t,B_t}(m,x)$ defined in (8).

We now derive the CDF of the pair (M_t^α, B_t^α) under the original measure $\widetilde{\mathbb{P}}$ by utilizing the change of measure argument. For any $m > 0$, $x \leq m$, using Girsanov's theorem,

$$\begin{aligned}\mathbb{P}(M_t^\alpha \leq m,\ B_t^\alpha \leq x) &= \mathbb{E}\left[\mathbb{1}_{\{M_t^\alpha \leq m,\ B_t^\alpha \leq x\}}\right] \\ &= \widetilde{\mathbb{E}}\left[\frac{1}{Z_t}\mathbb{1}_{\{M_t^\alpha \leq m,\ B_t^\alpha \leq x\}}\right]\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\mathbb{E}}\left[e^{\alpha B_t^\alpha - \frac{1}{2}\alpha^2 t} \mathbb{1}_{\{M_t^\alpha \le m,\ B_t^\alpha \le x\}}\right] \\
&= \int_0^m \int_{-\infty}^x e^{\alpha x - \frac{1}{2}\alpha^2 x} f_{M_t, B_t}(w, z)\, dz\, dw,
\end{aligned}
$$

where the last equality follows from the fact that the PDF of (M_t^α, B_t^α) is given by f_{M_t,B_t} under $\widetilde{\mathbb{E}}$. By differentiating both sides with respect to m followed by x, we obtain the desired result.

$\square$

We end this chapter by providing the unconditional CDF and PDF of M_t^α. By taking $\alpha = 0$, we immediately have the CDF of M_t.

Corollary 3.2 The unconditional CDF and PDF of M_t^α are respectively given by, for $m > 0$,

$$
\begin{aligned}
\mathbb{P}(M_t^\alpha \le m) &= N\left(\frac{m - \alpha t}{\sqrt{t}}\right) - e^{2\alpha m} N\left(-\frac{m + \alpha t}{\sqrt{t}}\right), \\
f_{M_t^\alpha}(m) &= \frac{2}{\sqrt{2\pi t}} e^{-\frac{(m-\alpha t)^2}{2t}} - 2\alpha e^{2\alpha m} N\left(-\frac{m + \alpha t}{\sqrt{t}}\right).
\end{aligned}
$$

Proof. We first derive the unconditional CDF. Integrating the joint density function (10), we have

$$
\begin{aligned}
\mathbb{P}(M_t^\alpha \le m) &= \int_{-\infty}^m \int_{x \vee 0}^m f_{M_t^\alpha, B_t^\alpha}(w, x)\, dw\, dx \\
&= \int_0^m \int_x^m f_{M_t^\alpha, B_t^\alpha}(w, x)\, dw\, dx + \int_{-\infty}^0 \int_0^m f_{M_t^\alpha, B_t^\alpha}(w, x)\, dw\, dx \\
&= -\int_0^m \frac{1}{\sqrt{2\pi t}} e^{\alpha x - \frac{1}{2}\alpha^2 t} e^{-\frac{(2w-x)^2}{2t}} \Big|_x^m dx - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi t}} e^{\alpha x - \frac{1}{2}\alpha^2 t} e^{-\frac{(2w-x)^2}{2t}} \Big|_0^m dx \\
&= \frac{1}{\sqrt{2\pi t}} \left[\int_0^m e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{x^2}{2t}} dx - \int_0^m e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{(2m-x)^2}{2t}} dx\right] \\
&\quad + \frac{1}{\sqrt{2\pi t}} \left[\int_{-\infty}^0 e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{x^2}{2t}} dx - \int_{-\infty}^0 e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{(2m-x)^2}{2t}} dx\right] \\
&= \frac{1}{\sqrt{2\pi t}} \left[\int_{-\infty}^m e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{x^2}{2t}} dx - \int_{-\infty}^m e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{(2m-x)^2}{2t}} dx\right].
\end{aligned}
$$

By completing squares and a change of variable,

$$
\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^m e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{x^2}{2t}} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^m e^{-\frac{(x-\alpha t)^2}{2t}} dx
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-\alpha t}{\sqrt{t}}} e^{-\frac{y^2}{2}}\, dy \\
&= N\left(\frac{m-\alpha t}{\sqrt{t}}\right).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\int_{-\infty}^{m} e^{\alpha x - \frac{1}{2}\alpha^2 t - \frac{(2m-x)^2}{2t}}\, dx &= \frac{e^{2\alpha m}}{\sqrt{2\pi t}} \int_{-\infty}^{m} e^{-\frac{(x-(2m+\alpha t))^2}{2t}}\, dx \\
&= \frac{e^{2\alpha m}}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{m+\alpha t}{\sqrt{t}}} e^{-\frac{y^2}{2}}\, dy \\
&= e^{2\alpha m} N\left(-\frac{m+\alpha t}{\sqrt{t}}\right).
\end{aligned}
$$

The CDF then follows by combining the above integrals.

To derive the PDF, we differentiate the CDF with respect to m:

$$
\begin{aligned}
f_{M_t^\alpha}(m) &= \frac{d}{dm}\mathbb{P}(M_t^\alpha \le m) \\
&= \frac{1}{\sqrt{t}} N'\left(\frac{m-\alpha t}{\sqrt{t}}\right) - 2\alpha e^{2\alpha m} N\left(-\frac{m+\alpha t}{\sqrt{t}}\right) + \frac{e^{2\alpha m}}{\sqrt{t}} N'\left(-\frac{m+\alpha t}{\sqrt{t}}\right) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(m-\alpha t)^2}{2t}} + \frac{e^{2\alpha m}}{\sqrt{t}} e^{-\frac{(m+\alpha t)^2}{2t}} - 2\alpha e^{2\alpha m} N\left(-\frac{m+\alpha t}{\sqrt{t}}\right) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(m-\alpha t)^2}{2t}} + \frac{1}{\sqrt{2\pi t}} e^{-\frac{(m-\alpha t)^2}{2t}} - 2\alpha e^{2\alpha m} N\left(-\frac{m+\alpha t}{\sqrt{t}}\right) \\
&= \frac{2}{\sqrt{2\pi t}} e^{-\frac{(m-\alpha t)^2}{2t}} - 2\alpha e^{2\alpha m} N\left(-\frac{m+\alpha t}{\sqrt{t}}\right).
\end{aligned}
$$

$\square$

Further Reading

1. Proof of the strong Markov property of Brownian motions;
2. The complete optional sampling theorem;
3. Reflected Brownian motions.