MATH5636 Stochastic Calculus for Finance II Spring 2026

Chapter 5: Pricing of Exotic Options: Barrier Options

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Preview

This chapter introduces methods for pricing path-dependent options whose payoffs depend on the ultimate maximum or minimum of the underlying risky asset. We shall look at two types of such options: a barrier option, which has a non-zero payoff only if the underlying asset price ever (or never) crosses a threshold; a lookback option, whose payoff depends on the maximum price of the underlying asset within certain period.

Key topics in this chapter:

1. Barrier options;
2. Lookback options.

1 Barrier Options

We shall focus on the Black-Scholes model throughout this chapter, i.e., the price of the risky asset is given by

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t),$$

or under the risk-neutral measure $\widetilde{\mathbb{P}}$,

$$dS_t = S_t(r\, dt + \sigma\, d\widetilde{B}_t). \tag{1}$$

The payoff of a barrier option is non-zero only if the price of the underlying asset *ever* or *never* hits a prescribed level, called the barrier. Barrier options can be classified into two main types, each of which has two subtypes:

1. **Knock-out options**: the option becomes worthless (i.e., is “knocked out”) if the price of the underlying asset crosses the barrier. The payoff is non-zero only if the barrier is *never* breached.
   (a) **Up-and-out options**: the option is knocked out if the asset price rises above a specified upper barrier.

   (b) **Down-and-out options**: the option is knocked out if the asset price falls below a specified lower barrier.

2. **Knock-in options**: the option becomes active (i.e., is "knocked in") only if the price of the underlying asset crosses the barrier at least once. The payoff is non-zero only if the barrier is *ever* breached.

   (a) **Up-and-in options**: the option comes into existence only if the asset price rises above a specified upper barrier. Otherwise, it pays nothing.

   (b) **Down-and-in options**: the option comes into existence only if the asset price falls below a specified lower barrier. Otherwise, it pays nothing.

In this section, we focus on up-and-out options. The pricing approach works analogously for the other three types of barrier options. The payoff of an up-and-out option with strike $K > 0$ and barrier $b > 0$ is given by

$$V_T = (S_T - K)^+ \mathbb{1}_{\left\{\max_{0\le t\le T} S_t \le b\right\}}.$$

Here, we assume that $K < b$ and $S_0 < b$; otherwise, the payoff of the option is always zero.

In the sequel, we consider two pricing approaches: the PDE approach by Black-Scholes equation, and the risk-neutral pricing approach.

1.1 Black-Scholes Equation

We begin by applying the risk-neutral pricing formula and aiming to derive a Markovian characterization of the option price using the Black-Scholes equation. Let $\{V_t\}_{t\in[0,T]}$ be the price of the option at time t. Using the risk-neutral pricing formula, we have

$$V_t = \widetilde{\mathbb{E}}[e^{-r(T-t)}V_T|\mathcal{F}_t] = e^{-r(T-t)}\widetilde{\mathbb{E}}\left[(S_T - K)^+ \mathbb{1}_{\left\{\max_{0\le t\le T} S_t \le b\right\}}|\mathcal{F}_t\right]. \tag{2}$$

We know that the discounted price process $\{e^{-rt}V_t\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale.

Recall that to apply the Black-Scholes PDE approach, the payoff must be Markovian with respect to the chosen state variables. However, the payoff of an up-and-out option depends on the random variable $\max_{0\le t\le T} S_t$, rather than solely on S_T. One way to restore the Markov property is to augment the state space by introducing a new process $S_t^* := \max_{0\le u\le t} S_u$, so that the payoff can be expressed as a function of the pair (S_T, S_T^*). This approach, however, involves the use of ***Brownian local time*** and the dynamics of S_t^*. The former is beyond the scope of this course, and we will discuss the latter in Lemma 2.1 below.

Alternatively, we present a method based on a stopping-time argument and condition on whether the barrier level has been breached. To this end, we define the stopping time τ_b, which is the first time the price of the underlying asset cross the barrier b:

$$\tau_b := \inf\{t \in [0, T] : S_t \ge b\} \wedge T.$$

Since $\{e^{-rt}V_t\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale, by the ***optional sampling theorem*** (Theorem 1.4, Chapter 4), the stopped process

$$e^{-r(t\wedge\tau_b)}V_{t\wedge\tau_b} = \begin{cases} e^{-rt}V_t, & \text{if } 0\le t\le \tau_b; \\ e^{-r\tau_b}V_{\tau_b}, & \text{if } \tau_b < t \le T, \end{cases}$$

is also a $\widetilde{\mathbb{P}}$-martingale.

The discounted price on the event $\{\tau_b < t \le T\}$ is straightforward. If $\tau_b < T$, the barrier has been breached, so the option is knocked out and $V_{\tau_b} = 0$. If $\tau_b = T$, the option survives until maturity, and the price at $t = T$ is simply the payoff.

Remark 1.1. For continuous-time diffusion such as geometric Brownian motion, the first hitting time of the barrier τ_b coincides almost surely with the knock-out event if $\tau_b < T$, since the process S_t almost surely exceeds the barrier in the interval $(\tau_b, \tau_b + \varepsilon)$ for any $\varepsilon > 0$. This is due to the strong Markov property together with ***Blumenthal's 0–1 law*** for Brownian motions, so the option is effectively knocked out at τ_b. If $\tau_b = T$, the option survives until maturity, since there is no remaining time for the underlying to cross the barrier. Indeed, $\tau_b = T \iff \max_{0\le t\le T} S_t \le b$.

The following theorem provides a Markovian characterization of the option price prior to barrier breach, expressed via the Black-Scholes PDE.

Theorem 1.2 Let $v : [0,T]\times[0,b] \to \mathbb{R}$ be the solution of the following PDE:

$$\begin{cases} v_t(t,s) + rsv_s(t,s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t,s) = rv(t,s), & (t,s)\in(0,T)\times(0,b); \\ v(t,0) = 0, & 0\le t\le T; \\ v(t,b) = 0, & 0 \le t < T; \\ v(T,s) = (s-K)^+, & 0\le s\le b. \end{cases} \tag{3}$$

Then, $\{e^{-r(t\wedge\tau_b)}v(t\wedge\tau_b, S_{t\wedge\tau_b})\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale. In addition, $V_t = v(t, S_t)$ for $0 \le t \le \tau_b$.

Proof. For any $t \le \tau_b$, the stock price process $S_t \le b$. In that case, $v(t, S_t)$ is well-defined since S_t stays in the spatial domain of v. Using (3) and applying Itô's lemma to the process $e^{-rt}v(t, S_t)$, we have, for $t \le \tau_b$,

$$\begin{aligned} d(e^{-rt}v(t,S_t)) &= e^{-rt}\left[-rv(t,S_t) + v_t(t,S_t) + rS_t v_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(t,S_t)\right] dt \\ &\quad + e^{-rt}\sigma S_t v_s(t,S_t)\, d\widetilde{B}_t \\ &= e^{-rt}\sigma S_t v_s(t,S_t)\, d\widetilde{B}_t. \end{aligned} \tag{4}$$

Integrating the above over the interval $[0, t \wedge \tau_b]$, we have

$$e^{-r(t\wedge\tau_b)} v(t \wedge \tau_b, S_{t\wedge\tau_b}) = v(0, S_0) + \int_0^{t\wedge\tau_b} e^{-ru} \sigma S_u v_s(u, S_u)\, d\widetilde{B}_u.$$

By the optional sampling theorem, the stopped Itô integral, $\int_0^{t\wedge\tau_b} e^{-ru}\sigma S_u v_s(u, S_u)\, d\widetilde{B}_u$, is a $\widetilde{\mathbb{P}}$-martingale, and so as $\{e^{-r(t\wedge\tau_b)} v(t\wedge\tau_b, S_{t\wedge\tau_b})\}_{t\in[0,T]}$.

On the other hand, integrating (4) from $t \wedge \tau_b$ to $T \wedge \tau_b$, we have

$$e^{-r(T\wedge\tau_b)} v(T \wedge \tau_b, S_{T\wedge\tau_b}) = e^{-r(t\wedge\tau_b)} v(t \wedge \tau_b, S_{t\wedge\tau_b}) + \int_{t\wedge\tau_b}^{T\wedge\tau_b} e^{-ru} \sigma S_u v_s(u, S_u)\, d\widetilde{B}_u.$$

Using the optional sampling theorem, we have

$$\begin{aligned}
& e^{-r(t\wedge\tau_b)} v(t \wedge \tau_b, S_{t\wedge\tau_b}) \\
&= \widetilde{\mathbb{E}}\left[e^{-r(T\wedge\tau_b)} v(T \wedge \tau_b, S_{T\wedge\tau_b}) \middle| \mathcal{F}_{t\wedge\tau_b}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-r(T\wedge\tau_b)} v(T \wedge \tau_b, S_{t\wedge\tau_b}) \mathbb{1}_{\{\tau_b<T\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] + \widetilde{\mathbb{E}}\left[e^{-r(T\wedge\tau_b)} v(T \wedge \tau_b, S_{t\wedge\tau_b}) \mathbb{1}_{\{\tau_b=T\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-r\tau_b} v(\tau_b, S_{\tau_b}) \mathbb{1}_{\{\tau_b<T\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] + \widetilde{\mathbb{E}}\left[e^{-rT} v(T, S_T) \mathbb{1}_{\{\tau_b=T\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-r\tau_b} v(\tau_b, b) \mathbb{1}_{\{\tau_b<T\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] + \widetilde{\mathbb{E}}\left[e^{-rT} v(T, S_T) \mathbb{1}_{\{\max_{0\le u\le T} S_u \le b\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT} (S_T - K)^+ \mathbb{1}_{\{\max_{0\le u\le T} S_u \le b\}} \middle| \mathcal{F}_{t\wedge\tau_b}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT} V_T \middle| \mathcal{F}_{t\wedge\tau_b}\right],
\end{aligned}$$

where we have used the fact that $\tau_b = T \iff \max_{0\le u\le T} S_u \le b$, and the boundary conditions of v specified in (3).

In particular, in the event $t \le \tau_b$, we have $t \wedge \tau_b = t$, and thus

$$e^{-rt} v(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-rT} V_T | \mathcal{F}_t\right], \ \{t \le \tau_b\}.$$

Combining with the risk-neutral pricing formula, $e^{-rt} V_t = \widetilde{\mathbb{E}}[e^{-rT} V_T | \mathcal{F}_t]$, we have, in the event $\{t \le \tau_b\}$,

$$e^{-rt} v(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-rT} V_T | \mathcal{F}_t\right] = e^{-rt} V_t,$$

so that $V_t = v(t, S_t)$ for $t \le \tau_b$. Finally, the condition $v(t, 0) = 0$ for $0 \le t \le T$ follows from the fact that once the geometric Brownian motion hits the level 0, it will stay at 0 afterwards. □

1.2 Risk-Neutral Pricing

In this subsection, we compute the risk-neutral price using (2) and the distributions of the Brownian motion and its running maximum.

To this end, upon solving the SDE (1), we have

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\widetilde{B}_t} = S_0 e^{\sigma\widetilde{B}_t^\alpha},$$

where

$$\alpha := \frac{r-\frac{\sigma^2}{2}}{\sigma} \quad \text{and} \quad \widetilde{B}_t^\alpha := \alpha t + \widetilde{B}_t.$$

Hence,

$$\max_{0\le t\le T} S_t = \max_{0\le t\le T} S_0 e^{\sigma\widetilde{B}_t^\alpha} = S_0 e^{\sigma\widetilde{M}_T^\alpha}, \tag{5}$$

where $\widetilde{M}_t^\alpha := \max_{0\le s\le t}\widetilde{B}_s^\alpha$.

Using these, we can rewrite the payoff V_T of the knock-out option as follows:

$$\begin{aligned} V_T &= \left(S_0 e^{\sigma\widetilde{B}_T^\alpha} - K\right)^+ \mathbb{1}_{\left\{S_0 e^{\sigma\widetilde{M}_T^\alpha}\le b\right\}} \\ &= \left(S_0 e^{\sigma\widetilde{B}_T^\alpha} - K\right) \mathbb{1}_{\left\{S_0 e^{\sigma\widetilde{B}_T^\alpha}\ge K,\ S_0 e^{\sigma\widetilde{M}_T^\alpha}\le b\right\}} \\ &= \left(S_0 e^{\sigma\widetilde{B}_T^\alpha} - K\right) \mathbb{1}_{\left\{\widetilde{B}_T^\alpha\ge k,\ \widetilde{M}_T^\alpha\le m\right\}}, \end{aligned} \tag{6}$$

where

$$k := \frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right),\ m := \frac{1}{\sigma}\ln\left(\frac{b}{S_0}\right).$$

Note that $m > 0$ and $m > k$ since $b > S_0$ and $b > K$. Hence,

$$V_t = e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\left(S_0 e^{\sigma\widetilde{B}_T^\alpha} - K\right)\mathbb{1}_{\left\{\widetilde{B}_T^\alpha\ge k,\ \widetilde{M}_T^\alpha\le m\right\}}\big|\mathcal{F}_t\right].$$

In what follows, we compute the time-0 price of the knock-out option by utilizing the joint distribution of $(\widetilde{M}_T^\alpha, \widetilde{B}_T^\alpha)$, a Brownian motion with drift and its running maximum, derived in the Chapter 4.

By Theorem 3.1 in Chapter 4, the joint density function of $(\widetilde{M}_T^\alpha, \widetilde{B}_T^\alpha)$ is given by

$$f_{\widetilde{M}_T^\alpha,\widetilde{B}_T^\alpha}(w,x) = \frac{2(2w-x)}{T\sqrt{2\pi T}} e^{\alpha x-\frac{1}{2}\alpha^2 T} e^{-\frac{(2w-x)^2}{2T}},\ x\le w,\ w>0.$$

To determine the integrating region, note that the density function is supported in $\{(w,x) : x \le w, w > 0\}$. In addition, the condition $\left\{\widetilde{B}_T^\alpha \ge k,\ \widetilde{M}_T^\alpha \le m\right\}$ restricts the integrating region to $\{(w,x) : k \le x,\ w \le m\}$. Combining the two regions, the integrating region is given by

$$\{(w,x) : k \le x \le m,\ x^+ \le w \le m\},$$

where $k^+ = \max\{k, 0\}$.

Using the integrating region, under $\widetilde{\mathbb{P}}$, the time-0 price of the knock-out option is given by

$$\begin{aligned}
V_0 &= e^{-rT}\widetilde{\mathbb{E}}\left[\left(S_0 e^{\sigma \widetilde{B}^\alpha_T} - K\right)\mathbb{1}_{\left\{\widetilde{B}^\alpha_T \ge k,\ \widetilde{M}^\alpha_T \le m\right\}}\right] \\
&= e^{-rT}\int_k^m \int_{x^+}^m (S_0 e^{\sigma x} - K)\, f_{\widetilde{M}^\alpha_T, \widetilde{B}^\alpha_T}(w,x)\, dw\, dx \\
&= e^{-rT}\int_k^m (S_0 e^{\sigma x} - K)\left(\int_{x^+}^m \frac{2(2w-x)}{T\sqrt{2\pi T}} e^{\alpha x - \frac{1}{2}\alpha^2 T} e^{-\frac{(2w-x)^2}{2T}}\, dw\right) dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_k^m (S_0 e^{\sigma x} - K) e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{x^2}{2T}}\, dx \\
&\quad - \frac{e^{-rT}}{\sqrt{2\pi T}}\int_k^m (S_0 e^{\sigma x} - K) e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{(2m-x)^2}{2T}}\, dx \\
&=: S_0 e^{-rT}(I_1 - I_2) - K e^{-rT}(I_3 - I_4). \qquad (7)
\end{aligned}$$

We define and compute the integrals $I_1, \ldots, I_4$ as follows. Consider integrals of the form

$$\begin{aligned}
\frac{1}{\sqrt{2\pi T}}\int_k^m e^{\gamma x - \frac{1}{2}\alpha^2 T - \frac{x^2}{2T}}\, dx &= e^{-\frac{(\alpha^2-\gamma^2)T}{2}}\int_k^m \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-\gamma T)^2}{2T}}\, dx \\
&= e^{-\frac{(\alpha^2-\gamma^2)T}{2}}\int_{\frac{k-\gamma T}{\sqrt{T}}}^{\frac{m-\gamma T}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\
&= e^{-\frac{(\alpha^2-\gamma^2)T}{2}}\left[N\left(\frac{m-\gamma T}{\sqrt{T}}\right) - N\left(\frac{k-\gamma T}{\sqrt{T}}\right)\right] \\
&= e^{-\frac{(\alpha^2-\gamma^2)T}{2}}\left[N\left(\frac{\ln\left(\frac{b}{S_0}\right) - \sigma\gamma T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(\frac{K}{S_0}\right) - \sigma\gamma T}{\sigma\sqrt{T}}\right)\right] \\
&= e^{-\frac{(\alpha^2-\gamma^2)T}{2}}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \sigma\gamma T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(\frac{S_0}{b}\right) + \sigma\gamma T}{\sigma\sqrt{T}}\right)\right], \qquad (8)
\end{aligned}$$

where we have used the fact that $N(-z) = 1 - N(z)$. Likewise, using (8) with γ being replaced with $\gamma + 2m/T$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi T}}\int_k^m e^{\gamma x - \frac{1}{2}\alpha^2 T - \frac{(2m-x)^2}{2T}}\, dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_k^m e^{\left(\gamma + \frac{2m}{T}\right)x - \frac{1}{2}\left(\alpha^2 T + \frac{4m^2}{T}\right) - \frac{x^2}{2T}}\, dx \\
&= e^{-\frac{1}{2}\left(\alpha^2 T + \frac{4m^2}{T}\right) + \frac{(\gamma + 2m/T)^2}{2}T}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right) + 2m\sigma + \sigma\gamma T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(\frac{S_0}{b}\right) + 2m\sigma + \sigma\gamma T}{\sigma\sqrt{T}}\right)\right]
\end{aligned}$$

$$= e^{-\frac{\alpha^2-\gamma^2}{2}T+2m\gamma}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+2\ln\left(\frac{b}{S_0}\right)+\sigma\gamma T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{S_0}{b}\right)+2\ln\left(\frac{b}{S_0}\right)+\sigma\gamma T}{\sigma\sqrt{T}}\right)\right]$$

$$= e^{-\frac{\alpha^2-\gamma^2}{2}T+2m\gamma}\left[N\left(\frac{\ln\left(\frac{b^2}{KS_0}\right)+\sigma\gamma T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{b}{S_0}\right)+\sigma\gamma T}{\sigma\sqrt{T}}\right)\right]. \tag{9}$$

To compute I_1, we use (8) with $\gamma = \sigma + \alpha$, which yields

$$\begin{aligned}
I_1 &:= \int_k^m \frac{1}{\sqrt{2\pi T}} e^{(\sigma+\alpha)x-\frac{1}{2}\alpha^2T-\frac{x^2}{2T}}\,dx \\
&= e^{-\frac{(\alpha^2-(\sigma+\alpha)^2)T}{2}}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\sigma(\sigma+\alpha)T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{S_0}{b}\right)+\sigma(\sigma+\alpha)T}{\sigma\sqrt{T}}\right)\right] \\
&= e^{\alpha\sigma+\frac{\sigma^2}{2}T}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\sigma(\sigma+\alpha)T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{S_0}{b}\right)+\sigma(\sigma+\alpha)T}{\sigma\sqrt{T}}\right)\right] \\
&= e^{rT}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{S_0}{b}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right] \\
&= e^{rT}\left[N\left(\delta_+\left(\frac{S_0}{K},T\right)\right)-N\left(\delta_+\left(\frac{S_0}{b},T\right)\right)\right], \qquad (10)
\end{aligned}$$

where

$$\delta_\pm(x,\tau) := \frac{\ln(x)+\left(r\pm\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \tag{11}$$

and we have used the definition of $\alpha = \frac{r-\frac{\sigma^2}{2}}{\sigma}$, so that $\alpha\sigma = r-\frac{\sigma^2}{2}$ and $\sigma(\sigma+\alpha) = r+\frac{1}{2}\sigma^2$.

Likewise, using (8) with $\gamma = \alpha$, we have

$$\begin{aligned}
I_3 &:= \int_k^m \frac{1}{\sqrt{2\pi T}} e^{(\alpha+\sigma)x-\frac{1}{2}\alpha^2T-\frac{x^2}{2T}}\,dx \\
&= e^{-\frac{\alpha^2-\alpha^2}{2}T}\left[N\left(\frac{\ln\left(\frac{S_0}{K}\right)+\sigma\alpha T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{S_0}{b}\right)+\sigma\alpha T}{\sigma\sqrt{T}}\right)\right] \\
&= N\left(\delta_-\left(\frac{S_0}{K},T\right)\right)-N\left(\delta_-\left(\frac{S_0}{b},T\right)\right). \qquad (12)
\end{aligned}$$

Next, we compute I_2 and I_4, which are of the form of (9). For I_2, by identifying $\gamma = \alpha+\sigma$, we have

$$I_2 := \frac{1}{\sqrt{2\pi T}}\int_k^m e^{\alpha x-\frac{1}{2}\alpha^2T-\frac{(2m-x)^2}{2T}}\,dx$$

$$\begin{aligned}
&= e^{-\frac{\alpha^2-(\alpha+\sigma)^2}{2}T+2m(\alpha+\sigma)}\left[N\left(\frac{\ln\left(\frac{b^2}{KS_0}\right)+\sigma(\alpha+\sigma)T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{b}{S_0}\right)+(\alpha+\sigma)T}{\sigma\sqrt{T}}\right)\right]\\
&= e^{rT}\left(\frac{b}{S_0}\right)^{\frac{2(\alpha+\sigma)}{\sigma}}\left[N\left(\delta_+\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_+\left(\frac{b}{S_0},T\right)\right)\right]\\
&= e^{rT}\left(\frac{b}{S_0}\right)^{1+\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_+\left(\frac{b}{S_0},T\right)\right)\right]. \qquad (13)
\end{aligned}$$

Likewise, to calculate I_4, we use (9) by identifying $\gamma = \alpha$, we have

$$\begin{aligned}
I_4 &:= \frac{1}{\sqrt{2\pi T}}\int_k^m e^{\alpha x-\frac{1}{2}\alpha^2T-\frac{(2m-x)^2}{2T}}\,dx\\
&= e^{2m\alpha}\left[N\left(\frac{\ln\left(\frac{b^2}{KS_0}\right)+\sigma\alpha T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{b}{S_0}\right)+\sigma\alpha T}{\sigma\sqrt{T}}\right)\right]\\
&= \left(\frac{b}{S_0}\right)^{\frac{2\alpha}{\sigma}}\left[N\left(\delta_-\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_-\left(\frac{b}{S_0},T\right)\right)\right]\\
&= \left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left[N\left(\delta_-\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_-\left(\frac{b}{S_0},T\right)\right)\right]. \qquad (14)
\end{aligned}$$

Therefore, substituting (10), (12), (13), (14) into (7), we obtain

$$\boxed{\begin{aligned}
V_0 = S_0&\left[N\left(\delta_+\left(\frac{S_0}{K},T\right)\right)-N\left(\delta_+\left(\frac{S_0}{b},T\right)\right)\right]\\
&- b\left(\frac{b}{S_0}\right)^{\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_+\left(\frac{b}{S_0},T\right)\right)\right]\\
&- Ke^{-rT}\left[N\left(\delta_-\left(\frac{S_0}{K},T\right)\right)-N\left(\delta_-\left(\frac{S_0}{b},T\right)\right)\right]\\
&+ Ke^{-rT}\left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left[N\left(\delta_-\left(\frac{b^2}{KS_0},T\right)\right)-N\left(\delta_-\left(\frac{b}{S_0},T\right)\right)\right].
\end{aligned}} \qquad (15)$$

For $t \in [0,T]$, by Theorem 1.2, the price of the option is $V_t = v(t,S_t)$ if $0 \le t \le \tau_b$, $V_t = 0$ if $\tau_b < T$ and $t > \tau_b$, and $V_T = (S_T-K)^+$. We can derive the formula of the function $v(t,s)$ as follows:

1. Case 1: $0 \le t \le T$, $s > B$: $v(t,s) = 0$ since the option knocks out;
2. Case 2: $0 \le t < T$, $s = B$: $v(t,s) = v(t,B) = 0$, since the option will immediately exceed the barrier b after t;

3. Case 3: $t = T$, $s \geq 0$: $v(t, s) = v(T, B) = (s - K)^+$, since the option expires and there is no additional time beyond T for the underlying asset price to exceed b;
4. Case 4: $0 \leq t \leq T, s = 0$: $v(t, s) = 0$ since once the geometric Brownian motion hits 0, it will stay at 0 afterwards;
5. Case 5: $0 \leq t < T$, $0 < s \leq B$: $v(t, s)$ can be obtained by replacing T with $\tau := T - t$, and S_0 with s, i.e.,

$$\boxed{\begin{aligned} V_t = s & \left[N\left(\delta_+\left(\frac{s}{K}, \tau\right)\right) - N\left(\delta_+\left(\frac{s}{b}, \tau\right)\right)\right] \\ & - b\left(\frac{b}{s}\right)^{\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\frac{b^2}{Ks}, \tau\right)\right) - N\left(\delta_+\left(\frac{b}{s}, \tau\right)\right)\right] \\ & - Ke^{-r\tau}\left[N\left(\delta_-\left(\frac{s}{K}, \tau\right)\right) - N\left(\delta_-\left(\frac{s}{b}, \tau\right)\right)\right] \\ & + Ke^{-r\tau}\left(\frac{s}{b}\right)^{1-\frac{2r}{\sigma^2}}\left[N\left(\delta_-\left(\frac{b^2}{Ks}, \tau\right)\right) - N\left(\delta_-\left(\frac{b}{s}, \tau\right)\right)\right]. \end{aligned}}$$

Note that Cases 1 – 3 are nothing but the boundary conditions for the PDE (3).

2 Lookback Options

The payoff of a ***floating strike lookback option*** depends on the ultimate maximum price of an underlying asset over the horizon $[0, T]$. Let

$$S_t^* := \max_{0 \leq u \leq t} S_u.$$

The payoff of a lookback option consider in this section is given by

$$V_T := S_T^* - S_T. \tag{16}$$

Note that the payoff is non-negative since $S_T^* \geq S_T$ by definition.

Using the risk-neutral pricing formula, the price V_t of the lookback option at $t \in [0, T]$ is given by

$$V_t = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S_T^* - S_T)|\mathcal{F}_t\right].$$

One can indeed show that the pair $\{(S_t, S_t^*)\}_{t \in [0,T]}$ is a Markov process (proof omitted herein), and thus there exists a function $v(t, s, y)$ such that

$$V_t = v(t, S_t, S_t^*).$$

2.1 Black-Scholes Equation

In this subsection, we shall derive the Black-Scholes pricing PDE for the lookback option. To this end, we need to examine the dynamical properties of the process S^*.

Lemma 2.1 The running maximum process S_t^* has zero quadratic variation, i.e., $\langle S^* \rangle_t = 0$. In addition, its differential satisfies $dS_t^* = 0$ whenever $S_t < S_t^*$.

Sketch of Proof. We first prove that $\langle S^* \rangle_t = 0$. Note that $S_t^* = \max_{0 \le u \le t} S_u$ has continuous sample paths since S is continuous. In addition, S_t^* is non-decreasing in t. Using these two facts, for any partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$, we have

$$\begin{aligned} \sum_{i=1}^{n} \left(S_{t_i}^* - S_{t_{i-1}}^* \right)^2 &\le \max_{j=1,\ldots,n} \left(S_{t_j}^* - S_{t_{j-1}}^* \right) \sum_{i=1}^{n} \left(S_{t_i}^* - S_{t_{i-1}}^* \right) \\ &= \max_{j=1,\ldots,n} \left(S_{t_j}^* - S_{t_{j-1}}^* \right) (S_T^* - S_0) \to 0 \end{aligned}$$

as $\|\Pi\| \to 0$, since $\max_{j=1,\ldots,n}(S_{t_j}^* - S_{t_{j-1}}^*) \to 0$.

We give a heuristic idea for the second statement. Since the running maximum S_t^* increases only when S_t reaches a new maximum. If $S_t < S_t^*$, continuity of S implies that this inequality persists locally in time, so S^* remains constant in a neighborhood of $[t-\delta, t]$ of t.[1] Hence, $dS_t^* = 0$ if $S_t < S_t^*$. □

We now provide the Black-Scholes pricing PDE for the lookback option.

Theorem 2.2 Let $v(t,s,y)$ denotes the price of the lookback option with payoff (16) at time $t \in [0,T]$ when $S_t = s$ and $S_t^* = y$. Then, v satisfies the following PDE:

$$\begin{cases} v_t(t,s,y) + rsv_s(t,s,y) + \dfrac{1}{2}\sigma^2 s^2 v_{ss}(t,s,y) = rv(t,s,y), & 0 \le t < T,\ 0 \le s \le y; \\ v(t,0,y) = e^{-r(T-t)}y, & 0 \le t \le T,\ y \ge 0; \\ v_y(t,y,y) = 0, & 0 \le t \le T,\ y > 0; \\ v(T,s,y) = y - s,\ 0 \le s \le y. \end{cases} \tag{17}$$

Examining (17), we observe that the PDE itself coincides with the standard Black–Scholes equation for vanilla options, despite the presence of an additional state variable. The distinction lies instead in the domain and the boundary conditions.

First, the effective domain of v is $\{(t,s,y) : t \in [0,T],\ 0 \le s \le y\}$, which reflects the fact that the running maximum S_t^* is always at least as large as the current stock price S_t. Second, the boundary condition $v(t,0,y) = e^{-r(T-t)}y$ follows from the observation that

---

[1] The choice of $\delta = \delta(\omega) > 0$ depends on the realization $S_t^*(\omega)$, which is thus random in nature.

once the stock price hits zero, it remains there, so the option payoff depends solely on the running maximum value at that time. Finally, the boundary condition $v_y(t, y, y) = 0$ ensures that variations in the running maximum do not contribute to the option value when $s = y$. Since the running maximum process increases only when $S_t = S_t^*$, this condition effectively eliminates the contribution of the dS_t^* term in the pricing argument; see the proof below.

Proof. It suffices to show that the process $\{e^{-rt}v(t, S_t, S_t^*)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale. In that case, using the terminal condition at $t = T$,

$$e^{-rt}v(t, S_t, S_t^*) = \widetilde{\mathbb{E}}[e^{-rT}v(T, S_T, S_T^*)|\mathcal{F}_t] = \widetilde{\mathbb{E}}[e^{-rT}V_T|\mathcal{F}_t] = e^{-rt}V_t,$$

where the last equality follows from the risk-neutral pricing formula. As such, $v(t, S_t, S_t^*) = V_t$ as desired.

To deduce the $\widetilde{\mathbb{P}}$-martingale property, we apply Itô's lemma to $\{e^{-rt}v(t, S_t, S_t^*)\}_{t\in[0,T]}$, and using the risk-neutral dynamics of S_t given by (1), we have

$$\begin{aligned}
&d\left(e^{-rt}v(t, S_t, S_t^*)\right) \\
&= -re^{-rt}v(t, S_t, S_t^*)\,dt + e^{-rt}v_t(t, S_t, S_t^*)\,dt + e^{-rt}v_s(t, S_t, S_t^*)\,dS_t + e^{-rt}v_y(t, S_t, S_t^*)\,dS_t^* \\
&\quad + \frac{1}{2}v_{ss}(t, S_t, S_t^*)\,d\langle S\rangle_t + v_{sy}(t, S_t, S_t^*)\,d\langle S, S^*\rangle_t + \frac{1}{2}v_{yy}(t, S_t, S_t^*)\,d\langle S^*\rangle_t \\
&= e^{-rt}\bigg(- rv(t, S_t, S_t^*) + v_t(t, S_t, S_t^*) + rv_s(t, S_t, S_t^*) + \frac{1}{2}\sigma^2 v_{ss}(t, S_t, S_t^*)\bigg)\,dt \\
&\quad + \sigma e^{-rt}v_s(t, S_t, S_t^*)\,d\widetilde{B}_t \\
&\quad + e^{-rt}v_y(t, S_t, S_t^*)\,dS_t^* + v_{sy}(t, S_t, S_t^*)d\langle S, S^*\rangle_t + \frac{1}{2}v_{yy}(t, S_t, S_t^*)\,d\langle S^*\rangle_t.
\end{aligned}$$

By Lemma 2.1, we have $d\langle S^*\rangle_t = 0$, which also implies $d\langle S, S^*\rangle_t = 0$. On the other hand, $dS_t^* \neq 0$ only on the event $\{S_t = S_t^*\}$. Hence,

$$\begin{aligned}
e^{-rt}v_y(t, S_t, S_t^*)\,dS_t^* &= e^{-rt}v_y(t, S_t, S_t^*)\mathbb{1}_{\{S_t=S_t^*\}}dS_t^* + e^{-rt}v_y(t, S_t, S_t^*)\mathbb{1}_{\{S_t<S_t^*\}}dS_t^* \\
&= e^{-rt}v_y(t, S_t^*, S_t^*)\mathbb{1}_{\{S_t=S_t^*\}}dS_t^* + e^{-rt}v_y(t, S_t, S_t^*)\mathbb{1}_{\{S_t<S_t^*\}}dS_t^* \\
&= 0.
\end{aligned}$$

Note that the first term vanishes because of the boundary condition $v_y(t, y, y) = 0$; and the second term is zero since $dS_t^* = 0$ on the event $\{S_t < S_t^*\}$.

Combining the above calculations and using the Black-Scholes PDE, we have

$$\begin{aligned}
&d\left(e^{-rt}v(t, S_t, S_t^*)\right) \\
&= e^{-rt}\bigg(- rv(t, S_t, S_t^*) + v_t(t, S_t, S_t^*) + rv_s(t, S_t, S_t^*) + \frac{1}{2}\sigma^2 v_{ss}(t, S_t, S_t^*)\bigg)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ \sigma e^{-rt} v_s(t, S_t, S_t^*)\, d\widetilde{B}_t \\
&= \sigma e^{-rt} v_s(t, S_t, S_t^*)\, d\widetilde{B}_t,
\end{aligned}
$$

which thus proves that $\{e^{-rt} v(t, S_t, S_t^*)\}_{t\in[0,T]}$ is a $\widetilde{\mathbb{P}}$-martingale.

$\square$

The PDE (17) involves two spatial variables, (s, y). We next introduce a dimension-reduction technique that allows us to transform it into an equivalent one-dimensional PDE. To this end, observe that for any $\lambda > 0$, the price of the option with payoff $\lambda S_T^* - \lambda S_T$ is exactly λ of the price of the option with payoff $S_T^* - S_T$. We thus deduce the following positive homogeneity of the function v:

$$
v(t, \lambda s, \lambda y) = \lambda v(t, s, y),\ \ \lambda > 0.
$$

Hence, for $t \in [0, T]$, $0 \le s \le y$, $y > 0$, we have

$$
v(t, s, y) = yv\left(t, \frac{s}{y}, 1\right) = yv(t, z, 1),
$$

where $z := s/y \in [0, 1]$. This motivates us to define the function $u : [0, T] \times [0, 1] \to \mathbb{R}$ by

$$
u(t, z) := v(t, z, 1). \tag{18}
$$

Hence, to solve v, one suffices to solve v, one suffices to solve u which satisfies a one-dimensional PDE as derived below.

Using the definition $v(t, s, y) = yu(t, z)$ with $z := s/y$, we have

$$
\frac{dz}{ds} = \frac{1}{y} \quad \text{and} \quad \frac{dz}{dy} = -\frac{s}{y^2} = -\frac{z}{y}.
$$

Hence,

$$
\begin{aligned}
v_t(t, s, y) &= yu_t(t, z),\ \ v_s(t, s, y) = yu_z(t, z)\frac{dz}{ds} = u_z(t, z), \\
v_y(t, s, y) &= u(t, z) + yu_z(t, z)\frac{dz}{dy} = u(t, z) - zu_z(t, z), \\
v_{ss}(t, s, y) &= u_{zz}(t, z)\frac{dz}{ds} = \frac{1}{y} u_{zz}(t, z).
\end{aligned} \tag{19}
$$

Substituting (19) into (17), we have

$$
\begin{aligned}
0 &= v_t(t, s, y) + rsv_s(t, s, y) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t, s, y) - rv(t, s, y) \\
&= yu_t(t, z) + rsu_z(t, z) + \frac{1}{2}\sigma^2 \left(\frac{s^2}{y}\right) u_{zz}(t, z) - ryu(t, z)
\end{aligned}
$$

$$= y\left[u_t(t,z) + rzu_z(t,z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t,z) - ru(t,z)\right],$$

and thus we arrive at

$$u_t(t,z) + rzu_z(t,z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z).$$

For the boundary conditions, when $s = 0$, $z = 0$ and thus

$$yu(t,0) = v(t,0,y) = e^{-r(T-t)}y \Rightarrow u(t,0) = e^{-r(T-t)}.$$

When $s = y$, we have $z = 1$, and thus

$$0 = v_y(t,y,y) = u_z(t,1) - u(t,1).$$

Finally, when $t = T$,

$$yu(t,z) = v(T,s,y) = y - s \Rightarrow u(t,z) = 1 - z.$$

Collecting the PDE and the boundary conditions, we see that u satisfies the following PDE:

$$\begin{cases} u_t(t,z) + rzu_z(t,z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t,z) = ru(t,z), & 0 \le t < T,\ 0 < z < 1; \\ u(t,0) = e^{-r(T-t)}, & 0 \le t \le T; \\ u_z(t,1) = u(t,1), & 0 \le t < T; \\ u(T,z) = 1 - z, & 0 \le z \le 1. \end{cases} \tag{20}$$

Note that u satisfies a one-dimensional parabolic PDE with a *Robin-type boundary condition* at $z = 1$.

2.2 Risk-Neutral Pricing

In this section, we provide an explicit formula for the lookback option using the risk-neutral pricing method and the joint distribution of $(\widetilde{M}_T^\alpha, \widetilde{B}_T^\alpha)$, where we recall that

$$\alpha = \frac{r - \frac{\sigma^2}{2}}{\sigma},\ \widetilde{B}_t^\alpha = \alpha t + \widetilde{B}_t,\ \widetilde{M}_t^\alpha = \max_{0 \le s \le t} \widetilde{B}_s^\alpha.$$

Recall from (5), under the risk-neutral probability measure $\widetilde{\mathbb{P}}$, we have, for any $0 \le t \le T$

$$S_T^* = S_0 e^{\sigma \widetilde{M}_T^\alpha} = S_0 e^{\sigma \widetilde{M}_t^\alpha} e^{\sigma\left(\widetilde{M}_T^\alpha - \widetilde{M}_t^\alpha\right)} = S_t^* e^{\sigma\left(\widetilde{M}_T^\alpha - \widetilde{M}_t^\alpha\right)}.$$

Note that for any $0 \le t \le T$,

$$\widetilde{M}_T^\alpha - \widetilde{M}_t^\alpha = \begin{cases} \max\limits_{t\le s\le T} \widetilde{B}_s^\alpha - \widetilde{M}_t^\alpha, & \text{if } \max\limits_{t\le s\le T} \widetilde{B}_s^\alpha > \widetilde{M}_t^\alpha; \\ 0, & \text{if } \max\limits_{t\le s\le T} \widetilde{B}_s^\alpha \le \widetilde{M}_t^\alpha. \end{cases}$$

Hence, we have

$$\begin{aligned} \widetilde{M}_T^\alpha - \widetilde{M}_t^\alpha &= \left[\max_{t\le s\le T} \widetilde{B}_s^\alpha - \widetilde{M}_t^\alpha\right]^+ \\ &= \left[\max_{t\le s\le T} \left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \left(\widetilde{M}_t^\alpha - \widetilde{B}_t^\alpha\right)\right]^+ \\ &= \left[\max_{t\le s\le T} \left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \frac{1}{\sigma}\log\left(\frac{S_0 e^{\sigma \widetilde{M}_t^\alpha}}{S_0 e^{\sigma \widetilde{B}_t^\alpha}}\right)\right]^+ \\ &= \frac{1}{\sigma}\left[\sigma \max_{t\le s\le T} \left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{S_t^*}{S_t}\right)\right]^+ \end{aligned}$$

Therefore, the risk-neutral price can also be represented as

$$\begin{aligned} V_t &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[S_T^* - S_T | \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[S_t^* e^{\sigma\left(\widetilde{M}_T^\alpha - \widetilde{M}_t^\alpha\right)} \big| \mathcal{F}_t\right] - e^{-r(T-t)}\widetilde{\mathbb{E}}\left[S_T | \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\widetilde{\mathbb{E}}\left[S_t^* \exp\left(\left[\sigma \max_{t\le s\le T}\left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{S_t^*}{S_t}\right)\right]^+\right) \big| \mathcal{F}_t\right] - e^{rt}\widetilde{\mathbb{E}}\left[e^{-rT} S_T | \mathcal{F}_t\right] \\ &= S_t^* e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\exp\left(\left[\sigma \max_{t\le s\le T}\left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{S_t^*}{S_t}\right)\right]^+\right) \big| \mathcal{F}_t\right] - S_t, \end{aligned} \tag{21}$$

where we have used the fact that $e^{-rt}S_t$ is a $\widetilde{\mathbb{P}}$-martingale in the last equality.

To compute the first expectation in (21), we note that S_t^*/S_t is $\mathcal{F}_t$-measurable, while $\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha$ is independent of $\mathcal{F}_t$, thanks to the independent increment of Brownian motions. Hence, by the independence lemma, we have

$$\widetilde{\mathbb{E}}\left[\exp\left(\left[\sigma \max_{t\le s\le T}\left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{S_t^*}{S_t}\right)\right]^+\right) \big| \mathcal{F}_t\right] = g(S_t, S_t^*),$$

where

$$g(s,y) := \widetilde{\mathbb{E}}\left[\exp\left(\left[\sigma \max_{t\le s\le T}\left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{y}{s}\right)\right]^+\right)\right], \ 0 \le s \le y. \tag{22}$$

Consequently,

$$V_t = S_t^* e^{-r(T-t)} g(S_t, S_t^*) - S_t.$$

The price of the lookback option thus admits a Markovian form $V_t = v(t, S_t, S_t^*)$, where

$$v(t, s, y) := ye^{-r(T-t)}g(s, y) - s.$$

To compute V_t, it suffices to compute the function $g(s, y)$. By the stationary increment of Brownian motions, we know that

$$\begin{aligned}
\max_{t\le s\le T} \widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha &= \max_{t\le s\le T}\left(\alpha(s-t) + \widetilde{B}_s - \widetilde{B}_t\right)\\
&\overset{d}{=} \max_{0\le s-t\le T-t}\left(\alpha(s-t) + \widetilde{B}_{s-t}\right)\\
&\overset{d}{=} \max_{0\le u\le T-t}\left(\alpha u + \widetilde{B}_u\right)\\
&= \widetilde{M}_{T-t}^\alpha = \widetilde{M}_\tau^\alpha,
\end{aligned}$$

where $\tau := T - t$. Hence,

$$\begin{aligned}
g(s, y) &= \widetilde{\mathbb{E}}\left[\exp\left(\left[\sigma \max_{t\le s\le T}\left(\widetilde{B}_s^\alpha - \widetilde{B}_t^\alpha\right) - \log\left(\frac{y}{s}\right)\right]^+\right)\right]\\
&= \widetilde{\mathbb{E}}\left[\exp\left(\left[\sigma\widetilde{M}_\tau^\alpha - \log\left(\frac{y}{s}\right)\right]^+\right)\right]\\
&= \widetilde{\mathbb{P}}\left(\widetilde{M}_\tau^\alpha \le \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right) + \frac{s}{y}\widetilde{\mathbb{E}}\left[e^{\sigma\widetilde{M}_\tau^\alpha}\mathbb{1}_{\left\{\widetilde{M}_\tau^\alpha > \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right\}}\right]. \qquad (23)
\end{aligned}$$

We proceed to compute the first probability in (23). Using the CDF of $\widetilde{M}_\tau^\alpha$ (see, Corollary 3.2 in Chapter 4), we have

$$\widetilde{\mathbb{P}}(\widetilde{M}_\tau^\alpha \le m) = N\left(\frac{m - \alpha\tau}{\sqrt{\tau}}\right) - e^{2\alpha m}N\left(-\frac{m + \alpha\tau}{\sqrt{\tau}}\right).$$

By taking $m = \frac{1}{\sigma}\log(y/s)$, we have

$$\begin{aligned}
N\left(\frac{m - \alpha\tau}{\sqrt{\tau}}\right) &= N\left(\frac{\frac{1}{\sigma}\log\left(\frac{y}{s}\right) - \alpha\tau}{\sqrt{\tau}}\right)\\
&= N\left(\frac{\log\left(\frac{y}{s}\right) - \sigma\alpha\tau}{\sigma\sqrt{\tau}}\right)\\
&= N\left(\frac{\log\left(\frac{y}{s}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)\\
&= N\left(-\frac{\log\left(\frac{s}{y}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)
\end{aligned}$$

$$= N\left(-\delta_{-}\left(\frac{s}{y},\tau\right)\right),$$

where $\delta_{\pm}$ was defined in (11). Likewise,

$$\begin{aligned} e^{2\alpha m} N\left(-\frac{m+\alpha\tau}{\sqrt{\tau}}\right) &= \left(\frac{y}{s}\right)^{\frac{2\alpha}{\sigma}} N\left(-\frac{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)+\alpha\tau}{\sqrt{\tau}}\right) \\ &= \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_{-}\left(\frac{y}{s},\tau\right)\right). \end{aligned}$$

Therefore, the first term in (23) is given by

$$\widetilde{\mathbb{P}}\left(\widetilde{M}_{\tau}^{\alpha} \le \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right) = N\left(-\delta_{-}\left(\frac{s}{y},\tau\right)\right) - \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_{-}\left(\frac{y}{s},\tau\right)\right). \tag{24}$$

Next, we compute the second term in (23). To this end, we make use of the density function of $\widetilde{M}_{\tau}^{\alpha}$ under $\widetilde{\mathbb{P}}$ (see, Corollary 3.2 in Chapter 4):

$$f_{\widetilde{M}_{\tau}^{\alpha}}(m) = \frac{2}{\sqrt{2\pi\tau}} e^{-\frac{(m-\alpha\tau)^2}{2\tau}} - 2\alpha e^{2\alpha m} N\left(-\frac{m+\alpha\tau}{\sqrt{\tau}}\right).$$

As such,

$$\begin{aligned} &\widetilde{\mathbb{E}}\left[e^{\sigma\widetilde{M}_{\tau}^{\alpha}} \mathbb{1}_{\left\{\widetilde{M}_{\tau}^{\alpha} > \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right\}}\right] \\ &= \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} e^{\sigma m} f_{\widetilde{M}_{\tau}^{\alpha}}(m)\, dm \\ &= \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{\sigma m - \frac{(m-\alpha\tau)^2}{2\tau}}\, dm - 2\alpha \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} e^{(2\alpha+\sigma)m} N\left(-\frac{m+\alpha\tau}{\sqrt{\tau}}\right) dm \end{aligned} \tag{25}$$

The first integral on the RHS of (25) can be computed as follows:

$$\begin{aligned} \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{\sigma m - \frac{(m-\alpha\tau)^2}{2\tau}}\, dm &= e^{\sigma\alpha\tau + \frac{\sigma^2\tau}{2}} \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{-\frac{(m-((\sigma+\alpha)\tau)^2}{2\tau}}\, dm \\ &= e^{r\tau} \int_{\frac{1}{\sqrt{\tau}}\left(\frac{1}{\sigma}\log\left(\frac{y}{s}\right)-(\sigma+\alpha)\tau\right)}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\ &= 2e^{r\tau} N\left(-\frac{1}{\sqrt{\tau}}\left(\frac{1}{\sigma}\log\left(\frac{y}{s}\right) - (\sigma+\alpha)\tau\right)\right) \\ &= 2e^{r\tau} N\left(\frac{\log\left(\frac{s}{y}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

$$= 2e^{r\tau} N\left(\delta_+\left(\frac{s}{y},\tau\right)\right). \tag{26}$$

Next, we calculate the second integral in (25). To this end, we shall rewrite $N(\cdot)$ as an integral, and apply Fubini's theorem to change the order of integration, depicted as follows:

$$\begin{aligned}
&\int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} e^{(2\alpha+\sigma)m} N\left(-\frac{m+\alpha\tau}{\sqrt{\tau}}\right) dm \\
&= \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} e^{(2\alpha+\sigma)m} \left(\int_{-\infty}^{-\frac{m+\alpha\tau}{\sqrt{\tau}}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz\right) dm \\
&= \int_{-\infty}^{-\frac{\sigma^{-1}\log(y/s)+\alpha\tau}{\sqrt{\tau}}} \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{-(\alpha\tau+z\sqrt{\tau})} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}+(2\alpha+\sigma)m}\, dm\, dz \\
&= \int_{-\infty}^{-\delta_-(y/s,\tau)} \int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{-(\alpha\tau+z\sqrt{\tau})} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}+\frac{2rm}{\sigma}}\, dm\, dz \\
&= \frac{\sigma}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)} e^{-\frac{z^2}{2}} e^{\frac{2rm}{\sigma}}\Big|_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{-(\alpha\tau+z\sqrt{\tau})} dz \\
&= \frac{\sigma}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)} \left(e^{-\frac{z^2}{2}-\frac{2r(\alpha\tau+z\sqrt{\tau})}{\sigma}} - \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} e^{-\frac{z^2}{2}}\right) dz.
\end{aligned} \tag{27}$$

Next, we compute the first integral in (27). By completing squares, we have

$$\begin{aligned}
\frac{\sigma}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)} e^{-\frac{z^2}{2}-\frac{2r(\alpha\tau+z\sqrt{\tau})}{\sigma}}\, dz &= \frac{\sigma e^{\frac{-2r\alpha\tau}{\sigma}+\frac{2r^2\tau}{\sigma^2}}}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)} e^{-\frac{(z+\frac{2r\sqrt{\tau}}{\sigma})^2}{2}}\, dz \\
&= \frac{\sigma e^{r\tau}}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)} e^{-\frac{(z+\frac{2r\sqrt{\tau}}{\sigma})^2}{2}}\, dz \\
&= \frac{\sigma e^{r\tau}}{2r\sqrt{2\pi}} \int_{-\infty}^{-\delta_-(y/s,\tau)+\frac{2r\sqrt{\tau}}{\sigma}} e^{-\frac{\xi^2}{2}}\, d\xi \\
&= \frac{\sigma e^{r\tau}}{2r\sqrt{2\pi}} \int_{-\infty}^{\delta_+(s/y,\tau)} e^{-\frac{\xi^2}{2}}\, d\xi \\
&= \frac{\sigma e^{r\tau}}{2r} N\left(\delta_+\left(\frac{s}{y},\tau\right)\right).
\end{aligned} \tag{28}$$

On the other hand, the second integral in (27) can be computed as follows:

$$\frac{\sigma}{2r\sqrt{2\pi}} \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} \int_{-\infty}^{-\delta_-(y/s,\tau)} e^{-\frac{z^2}{2}}\, dz = \frac{\sigma}{2r} \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\frac{y}{s},\tau\right)\right). \tag{29}$$

Therefore, substituting (28) and (29) into (27) yields

$$\int_{\frac{1}{\sigma}\log\left(\frac{y}{s}\right)}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{\sigma m - \frac{(m-\alpha\tau)^2}{2\tau}}\, dm = \frac{\sigma e^{r\tau}}{2r} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) - \frac{\sigma}{2r}\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right). \tag{30}$$

Consequently, we can substitute (26) (30) into (25) to obtain

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left[e^{\sigma \widetilde{M}_\tau^\alpha} \mathbb{1}_{\left\{\widetilde{M}_\tau^\alpha > \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right\}}\right] \\
&= 2e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) - \frac{\sigma\alpha e^{r\tau}}{r} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) + \frac{\sigma\alpha}{r}\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right) \\
&= 2e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) - \left(1 - \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right)\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right) \\
&= \left(1 + \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) + \left(1 - \frac{\sigma^2}{2r}\right)\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right).
\end{aligned} \tag{31}$$

Therefore, substituting (24) and (31) into (23), we obtain

$$\begin{aligned}
g(s,y) &= \widetilde{\mathbb{P}}\left(\widetilde{M}_\tau^\alpha \le \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right) + \frac{s}{y}\widetilde{\mathbb{E}}\left[e^{\sigma \widetilde{M}_\tau^\alpha} \mathbb{1}_{\left\{\widetilde{M}_\tau^\alpha > \frac{1}{\sigma}\log\left(\frac{y}{s}\right)\right\}}\right] \\
&= N\left(-\delta_-\left(\frac{s}{y}, \tau\right)\right) - \left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right) + \frac{s}{y}\left(1 + \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right)\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right) \\
&= \frac{s}{y}\left(1 + \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) + N\left(-\delta_-\left(\frac{s}{y}, \tau\right)\right) \\
&\quad - \frac{\sigma^2}{2r}\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}-1} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right).
\end{aligned}$$

Finally, for $0 \le t \le T$, $0 < s \le y$, and $\tau = T - t$, the price of the lookback option, $v(t,s,y)$, is given by

$$\boxed{\begin{aligned}
v(t,s,y) &= ye^{-r\tau} g(s,y) - s \\
&= s\left(1 + \frac{\sigma^2}{2r}\right) N\left(\delta_+\left(\frac{s}{y}, \tau\right)\right) + ye^{-r\tau} N\left(-\delta_-\left(\frac{s}{y}, \tau\right)\right) \\
&\quad - \frac{\sigma^2}{2r} s\left(\frac{y}{s}\right)^{\frac{2r}{\sigma^2}} e^{-r\tau} N\left(-\delta_-\left(\frac{y}{s}, \tau\right)\right) - s.
\end{aligned}}$$

Further Readings

1. Reflected Brownian motions;
2. Brownian local time;
3. Pricing of American options.